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## Power Geometry as new Mathematics

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Abstract			

Here we present basic ideas and algorithms of Power Geometry and give a survey of some of its applications. In Section 1, we consider one generic ordinary differential equation and demonstrate how to find asymptotic forms and asymptotic expansions of its solutions. In Section 2, we demonstrate how to find expansions of solutions to Painlevé equations by this method, and we analyze singularities of plane oscillations of a satellite on an elliptic orbit. In Section 3, we expound the spacial generalizations of planar constructions. Power Geometry gives alternatives to Algebraic Geometry, Differential Algebra, Nonstandard Analysis, Microlocal Analysis and Group Analysis.

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Introduction			

Elements of plane PG were proposed by Newton for algebraic equations (1680); and by Briot and Bouquet for ordinary differential equations (1856). Space PG for a nonlinear autonomous system of ODEs were proposed by the author (1962), and for a linear PDE, by Mikhailov (1963).

In this talk we intend to give basic notions of PG, present some of its algorithms, results, and applications. It is clear that this calculus cannot be mastered during this presentation.

The calculus is subject for one-year course of lectures in Moscow State University.

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Statement of the pr	roblem		

First, consider one differential equation and power-logarithmic expansions of its solutions (although there possible more complex expansions).

Let x be independent and y be dependent variables,  $x, y \in \mathbb{C}$ . A differential monomial a(x, y) is a product of an ordinary monomial  $cx^{r_1}y^{r_2}$ , where  $c = \text{const} \in \mathbb{C}$ ,  $(r_1, r_2) \in \mathbb{R}^2$ , and a finite number of derivatives of the form  $d^l y/dx^l$ ,  $l \in \mathbb{N}$ . A sum of differential monomials

$$f(x,y) = \sum a_i(x,y) \tag{1.1}$$

is called the *differential sum*.

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The main problem			

Let a differential equation be given

$$f(x,y) = 0,$$
 (1.2)

where f(x, y) is a differential sum. As  $x \to 0$ , or as  $x \to \infty$ , for solutions  $y = \varphi(x)$  to the equation (1.2), find all expansions of the form

$$y = c_r x^r + \sum c_s x^s, \quad c_r = \text{const} \in \mathbb{C}, \quad c_r \neq 0,$$
 (1.3)

where  $c_s$  are polynomials in log x, and power exponents  $r, s \in \mathbb{R}$ ,

$$\omega r > \omega s,$$
 (1.4)

$$\omega = -1$$
, if  $x \to 0$ ,  $\omega = 1$ , if  $x \to \infty$ . (1.5)

The procedure to compute expansions (1.3) consists of two steps: computation of the first approximations

$$y = c_r x^r, \quad c_r \neq 0 \tag{1.6}$$

and computation of further expansion terms in (1.3).

Plane PG. Theory Plane PG. Applications Space PG Other apps of PG Computation of truncated equations To each differential monomial a(x, y), we assign its (vector) power exponent  $Q(a) = (q_1, q_2) \in \mathbb{R}^2$  by the following rules:  $Q(cx^{r_1}v^{r_2}) = (r_1, r_2); \quad Q(d^l v/dx^l) = (-l, 1);$ when differential monomials are multiplied, their power exponents must be added as vectors  $Q(a_1a_2) = Q(a_1) + Q(a_2)$ . The set S(f) of power exponents  $Q(a_i)$  of all differential monomials  $a_i(x, y)$  present in the differential sum (1.1) is called the support of the sum f(x, y). Obviously,  $\mathbf{S}(f) \in \mathbb{R}^2$ . The convex hull  $\Gamma(f)$  of the support S(f) is called the *polygon of the sum* f(x, y). The boundary  $\partial \Gamma(f)$  of the polygon  $\Gamma(f)$  consists of the vertices  $\Gamma_i^{(0)}$ and the edges  $\Gamma_i^{(1)}$ . They are called (generalized) faces  $\Gamma_i^{(d)}$ , where the upper index indicates the dimension of the face, and the lower

one is its number. Each face  $\Gamma_i^{(d)}$  corresponds to the *truncated sum* 

$$\hat{f}_j^{(d)}(x,y) = \sum a_i(x,y) \text{ over } Q(a_i) \in \Gamma_j^{(d)} \cap \mathbf{S}(f).$$
 (1.7)

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Example			
Consider the third	d Painlevé equation		
$f(x,y) \stackrel{\mathrm{def}}{=} -xy$	$y'' + xy'^2 - yy' + ay^3 + b$	$y + cxy^4 + dx = 0$	0, (1.8)

assuming the complex parameters  $a, b, c, d \neq 0$ . Here the first three differential monomials have the same power exponent  $Q_1 = (-1, 2)$ , then  $Q_2 = (0, 3)$ ,  $Q_3 = (0, 1)$ ,  $Q_4 = (1, 4)$ ,  $Q_5 = (1, 0)$ . They are shown in Fig. 1 in coordinates  $q_1, q_2$ .



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Example			

Their convex hull  $\Gamma(f)$  is the triangle with three vertices  $\Gamma_1^{(0)} = Q_1$ ,  $\Gamma_2^{(0)} = Q_4$ ,  $\Gamma_3^{(0)} = Q_5$ , and with three edges  $\Gamma_1^{(1)}$ ,  $\Gamma_2^{(1)}$ ,  $\Gamma_3^{(1)}$ . The vertex  $\Gamma_1^{(0)} = Q_1$  corresponds to the truncation

$$\hat{f}_1^{(0)}(x,y) = -xyy'' + xy'^2 - yy',$$

and the edge  $\Gamma_1^{(1)}$  corresponds to the truncation

$$\hat{f}_1^{(1)}(x,y) = \hat{f}_1^{(0)}(x,y) + by + dx.$$

Let the plane  $\mathbb{R}^2_*$  be dual to the plane  $\mathbb{R}^2$  such that for  $P = (p_1, p_2) \in \mathbb{R}^2_*$  and  $Q = (q_1, q_2) \in \mathbb{R}^2$ , the scalar product

$$\langle P, Q \rangle \stackrel{\mathrm{def}}{=} p_1 q_1 + p_2 q_2$$

is defined. Each face  $\Gamma_j^{(d)}$  in  $\mathbb{R}^2_*$  corresponds to its own *normal cone*  $\mathbf{U}_j^{(d)}$  formed by the outward normal vectors P to the face  $\Gamma_j^{(d)}$ . For the edge  $\Gamma_j^{(1)}$ , the normal cone  $\mathbf{U}_j^{(1)}$  is the ray orthogonal to the edge  $\Gamma_j^{(1)}$  and directed outward the polygon  $\Gamma(f)$ . For the vertex  $\Gamma_j^{(0)}$ , the normal cone  $\mathbf{U}_j^{(0)}$  is the open sector (angle) in the plane  $\mathbb{R}^2_*$  with the vertex at the origin P = 0 and limited by the rays which are the normal cones of the edges adjacent to the vertex  $\Gamma_j^{(0)}$ .

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xample			
For the the equatio normal cones $U_j^{(d)}$ $\Gamma_j^{(d)}$ are shown in F	n (1.8), the of the faces ïg. 2. ■	$U_{2}^{(4)}$ $P_{2}$ $U_{1}^{(0)}$ $0$ $U_{1}^{(4)}$ $0$	$v_{2}^{(0)}$ $v_{3}^{(1)}$ $P_{1}$ $v_{3}^{(0)}$

P

F

Plane PG. Theory Plane PG. Applications Space PG Other apps of PG Theorem 1 Thus, each face  $\Gamma_i^{(d)}$  corresponds to the normal cone  $\mathbf{U}_i^{(d)}$  in the plane  $\mathbb{R}^2_*$  and to the truncated equation  $\hat{f}_i^{(d)}(x,y) = 0.$ (1.9)Theorem (1) If the expansion (1.3) satisfies the equation (1.2), and  $\omega(1,r) \in U_i^{(d)}$ , then the truncation  $y = c_r x^r$  of the solution (1.3) is the solution to the truncated equation  $\hat{f}_i^{(d)}(x, y) = 0$ .

Hence, to find all truncated solutions  $y = c_r x^r$  to the equation (1.2), we need to compute: the support  $\mathbf{S}(f)$ , the polygon  $\Gamma(f)$ , all its faces  $\Gamma_j^{(d)}$ , and their normal cones  $\mathbf{U}_j^{(d)}$ . Then for each truncated equation  $\hat{f}_j^{(d)}(x, y) = 0$ , we need to find all its solutions  $y = c_r x^r$  which have one of the vectors  $\pm (1, r)$  lying in the normal cone  $\mathbf{U}_j^{(d)}$ .

The vertex  $\Gamma_j^{(0)} = \{Q\}$  corresponds to the truncated equation  $\hat{f}_j^{(0)}(x, y) = 0$  the support of which consists of one point  $Q = (q_1, q_2)$ . Take  $g(x, y) = x^{-q_1}y^{-q_2}\hat{f}_j^{(0)}(x, y)$ , then  $g(x, cx^r)$  does not depend on x and c, and it is a polynomial in r. Consequently, for the solution  $y = c_r x^r$  to the equation  $\hat{f}_j^{(0)}(x, y) = 0$ , the power exponent r is the root of characteristic equation

$$\chi(r) \stackrel{\text{def}}{=} g(x, x^r) = 0, \qquad (1.10)$$

with an arbitrary coefficient  $c_r$ . We need only those roots r of the equation (1.10) for which the vector  $\omega(1, r)$  lies in the normal cone  $\mathbf{U}_j^{(0)}$  of the vertex  $\Gamma_j^{(0)}$ .

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Example			

For the equation (1.8), the vertex  $\Gamma_1^{(0)} = Q_1 = (-1, 2)$  corresponds to the truncated equation

$$\hat{f}_{1}^{(0)}(x,y) \stackrel{\text{def}}{=} -xyy'' + x{y'}^2 - yy' = 0,$$
 (1.11)

and  $\hat{f}_1^{(0)}(x, x^r) = x^{2r-1}[-r(r-1) + r^2 - r] \equiv 0$ , i.e. any expression  $y = cx^r$  is a solution to the equation (1.11). Here  $\omega = -1$ , and we are interested only in those solutions which have the vector  $-(1, r) \in U_1^{(0)}$ . According to Fig. 2, this means that  $r \in (-1, 1)$ . Thus, the vertex  $\Gamma_1^{(0)}$  corresponds to the two-parameter family of power asymptotic forms of solutions

$$y = cx^r$$
, arbitrary  $c \neq 0$ ,  $r \in (-1, 1)$ .

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The edge  $\Gamma_j^{(1)}$  corresponds to the truncated equation  $\hat{f}_j^{(1)}(x, y) = 0$ , the normal cone  $\mathbf{U}_j^{(1)}$  of the edge is the ray  $\{P = \lambda \omega'(1, r'), \lambda > 0\}$ . The inclusion  $\omega(1, r) \in \mathbf{U}_j^{(1)}$  means the equalities  $\omega = \omega'$  and r = r'. This determines uniquely the power exponent r of the truncated solution  $y = c_r x^r$  and the value  $\omega$ . To determine the coefficient  $c_r$ , we need to substitute the expression  $y = c_r x^r$  into the truncated equation  $\hat{f}_j^{(1)}(x, y) = 0$ . After cancelation of some power of x, we obtain an algebraic equation for the coefficient  $c_r$ 

$$\tilde{\tilde{f}}(c_r) \stackrel{\text{def}}{=} x^{-s} \hat{f}_j^{(1)}(x, c_r x^r) = 0.$$
 (1.13)

Each root  $c_r \neq 0$  of this equation corresponds to its own asymptotic form  $y = c_r x^r$ .

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Example			

For the equation (1.8), the edge  $\Gamma_1^{(1)}$  corresponds to the truncated equation

$$\hat{f}_1(x,y) \stackrel{\text{def}}{=} -xyy'' + x{y'}^2 - yy' + by + dx = 0.$$
 (1.14)

Since  $U_1^{(1)} = \{P = -\lambda(1, 1), \lambda > 0\}$ , then  $\omega = -1$  and r = 1. Substituting  $y = c_1 x$  into the truncated equation (1.14) and canceling x, we obtain the equation  $bc_1 + d = 0$  for  $c_1$ , whence  $c_1 = -d/b$ . Thus, the edge  $\Gamma_1^{(1)}$  corresponds to a unique power asymptotic form of solutions

$$y = -(d/b)x. \quad \blacksquare \tag{1.15}$$

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The truncated equation  $\hat{f}_j^{(d)}(x, y) = 0$  may have non-power solutions  $y = \varphi(x)$  which are the asymptotic forms for solutions to the original equation f(x, y) = 0. These non-power solutions  $y = \varphi(x)$  may be found using power and logarithmic transformations. *Power transformation* is linear in logarithms

$$\begin{split} &\log x = \alpha_{11} \log u + \alpha_{12} \log v, \\ &\log y = \alpha_{21} \log u + \alpha_{22} \log v, \end{split} \quad \alpha = \left( \begin{array}{cc} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{array} \right), \alpha_{ij} \in \mathbb{R}, \ \det \alpha \neq 0. \end{split}$$

It induces linear dual transformations in spaces  $\mathbb{R}^2$  and  $\mathbb{R}^2_*$ . Logarithmic transformation has the form

$$\xi = \log u \text{ or } \eta = \log v.$$

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Example			

For the truncated equation (1.14) corresponding to the edge  $\Gamma_1^{(1)}$  with the normal vector -(1, 1), we make power transformation

$$\log x = \log u \log y = \log u + \log v, \quad \alpha = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix},$$

i.e. x = u, y = uv. Since y' = xv' + v, y'' = xv'' + 2v', then, canceling x and collecting similar terms, the equation (1.14), takes the form

$$-x^{2}vv'' + x^{2}v'^{2} - xvv' + bv + d = 0.$$
 (1.16)

Its support consists of three points  $\tilde{Q}_1 = (0,2)$ ,  $\tilde{Q}_2 = (0,1)$ ,  $\tilde{Q}_3 = 0$  placed on the axis  $\tilde{q}_1 = 0$ . Now we make the logarithmic transformation  $\xi = \log x$ .

Since  $v' = \dot{v}/x$ ,  $v'' = (\ddot{v} - \dot{v})/x^2$ , where  $\dot{=} d/d\xi$ , then, collecting similar terms, the equation (1.16) takes the form

$$-v\ddot{v}+\dot{v}^2+bv+d=0.$$

Applying the technique described below to this equation, we obtain the expansion of its solutions

$$v=-(b/2)\xi^2+\tilde{c}\xi+\sum_{k=0}^{\infty}c_k\xi^{-k},$$

where  $\tilde{c}$  is an arbitrary constant, and the constants  $c_k$  are uniquely determined. In original variables, we obtain the family of non-power asymptotic forms of solutions to the original equation (1.8)

$$y \sim x \left[-(b/2)(\log x)^2 + \tilde{c} \log x + \sum_{k=0}^{\infty} c_k (\log x)^{-k}\right], \ x \to 0.$$

Plane PG. Applications Space PG Plane PG. Theory Other apps of PG Critical numbers of a truncated solution If a truncated solution  $y = c_r x^r$  is found, then the substitution

 $y = c_r x^r + z$  reduces the equation f(x, y) = 0 to the form

$$f(x,cx^{r}+z) \stackrel{\text{def}}{=} \tilde{f}(x,z) \stackrel{\text{def}}{=} \mathcal{L}(x)z + h(x,z) = 0, \qquad (1.17)$$

where  $\mathcal{L}(x)$  is a linear differential operator, and the support  $\mathbf{S}(\mathcal{L}z)$ consists of only one point (v, 1) that is the vertex  $\tilde{\Gamma}_{1}^{(0)}$  of the polygon  $\Gamma(\tilde{f})$ ; the point (v, 1) is not in the support **S**(*h*). The operator  $\mathcal{L}(x)$  is computed as the first variation  $\delta \hat{f}_i^{(d)} / \delta y$  on the curve  $y = c_r x^r$ . Let  $\nu(k)$  be characteristic polynomial of the differential sum  $\mathcal{L}(x)z$ , i.e.

$$\nu(k) = x^{-\nu-k} \mathcal{L}(x) x^k. \tag{1.18}$$

The real roots  $k_1, \ldots, k_{\varkappa}$  of the polynomial  $\nu(k)$  that satisfy the inequality  $\omega r > \omega k_i$  are called the *critical numbers of the truncated* solution  $y = c_r x^r$ .

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Example			

For the truncated equation (1.11), the first variation is

$$\frac{\delta \hat{f}_1^{(0)}}{\delta y} = -xy'' - xy\frac{d^2}{dx^2} + 2xy'\frac{d}{dx} - y' - y\frac{d}{dx}$$

On the curve  $y = c_r x^r$ , this variation gives the operator

$$\mathcal{L}(x) = c_r x^{r-1} \left[ -r(r-1) - x^2 \frac{d^2}{dx^2} + 2rx \frac{d}{dx} - r - x \frac{d}{dx} \right].$$

The characteristic polynomial of the sum  $\mathcal{L}(x)z$ , i.e.  $\mathcal{L}(x)x^k$ , is

$$\nu(k) = c_r[-r(r-1) - k(k-1) + 2rk - r - k] = -c_r(k-r)^2.$$

It has one double root  $k_1 = r$ , which is not a critical number, since it does not satisfy the inequality  $\omega r > \omega k_1$ . Consequently, truncated solutions (1.12) have no critical numbers.

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For the truncated equation (1.14), the first variation is

$$\frac{\delta \hat{f}^{(1)}}{\delta y} = \frac{\delta \hat{f}_1^{(0)}}{\delta y} + b.$$

On the curve (1.15), i.e.  $y = c_1 x$ ,  $c_1 = -d/b$ , this variation gives the operator

$$\mathcal{L}(x) = c_1 \left[ -x^2 \frac{d^2}{dx^2} + 2x \frac{d}{dx} - 1 - x \frac{d}{dx} - \frac{b^2}{d} \right]$$

and the characteristic polynomial

$$\nu(k) = -c_1[k^2 - 2k + 1 + b^2/d].$$

Its roots are  $k_{1,2} = 1 \pm b/\sqrt{-d}$ . If  $\text{Im}(b/\sqrt{-d}) \neq 0$ , then real critical numbers are absent. If  $\text{Im}(b/\sqrt{-d}) = 0$ , then the inequality  $\omega r > \omega k_i$  is satisfied by only one root  $k_1 = 1 + |b/\sqrt{-d}|$  which is a unique critical number of the power asymptotic form (1.15).

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 Computation of asymptotic expansion (1.3)

Using support  $S(\tilde{f})$  of the equation (1.17) and numbers  $k_1, \ldots, k_{\varkappa}$  with  $\omega r > \omega k_i$ , we can find the set of numbers  $K(k_1, \ldots, k_{\varkappa}) \subset \mathbb{R}$ . Its elements *s* satisfy the inequality  $\omega r > \omega s$ .

#### Theorem (2)

The equation (1.17) has an expansion of solutions of the form

$$z = \sum c_s(\log x) x^s \text{ over } s \in \mathsf{K}(k_1, \dots, k_\varkappa), \tag{1.19}$$

where  $k_1, \ldots, k_{\varkappa}$  are critical numbers of the truncated solution  $y = c_r x^r$ ;  $c_s$  are polynomials in  $\log x$ , which are uniquely defined for  $s \neq k_i$ . If all critical numbers  $k_1, \ldots, k_{\varkappa}$  are simple roots, and each  $k_i$  does not lie in the set  $K(k_1, \ldots, k_{i-1}, k_{i+1}, \ldots, k_{\varkappa})$ , then all coefficients  $c_s$  are constant; for  $s \neq k_i$ , they are uniquely determined; and for  $s = k_i$ , they are arbitrary.

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Example			

For the truncated solution (1.12)

$$\mathbf{K} = \{ s = r + l(1-r) + m(1+r), \text{ int. } l, m \ge 0, l+m > 0 \}.$$
 (1.20)

Since there are no critical numbers, then all  $c_s$  are constant and uniquely determined in the expansion (1.19).

For the truncated solution (1.15)  $\mathbf{K} = \{s = 1 + 2l, \text{ int. } l > 0\}$ . If  $\operatorname{Im}(b/\sqrt{-d}) \neq 0$ , then there are no critical numbers, and all power exponents s are odd integers greater than 1 in the expansion (1.19), and coefficients  $c_s$  are constant and uniquely determined. If  $\operatorname{Im}(b/\sqrt{-d}) = 0$ , then there is a unique critical number  $k_1 = 1 + |b/\sqrt{-d}|$ , and

$$\mathbf{K}(k_1) = \{s = 1 + 2l + m(k_1 - 1), \text{ int. } l, m \ge 0, l + m > 0.\} \quad (1.21)$$

Consequently, if the number  $k_1$  is not odd, then all  $c_s$  are constant and uniquely determined in the expansion (1.19) for  $s \neq k_1$ , and  $c_{k_1}$  is arbitrary. Finally, if  $k_1$  is odd, then  $\mathbf{K}(k_1) = \mathbf{K}$ , and  $c_s$  is a uniquely determined constant in the expansion (1.19) if  $s < k_1$ ;  $c_{k_1}$  is a linear function of log x with an arbitrary constant term;  $c_s$  is a uniquely determined polynomial in log x if  $s > k_1$ .

Complex power expo	onents		
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Expansions of solutions (1.3) with complex power exponents r and s, where  $\omega \operatorname{Re} r > \omega \operatorname{Re} s$ , are found in a similar way.

#### Example

In the equation (1.8), for the truncated solution (1.12) with complex r,  $\operatorname{Re} r \in (-1, 1)$ , the expansions (1.19) are also found by the set (1.20). And for the truncated solution (1.15) with  $\operatorname{Im} (b/\sqrt{-d}) \neq 0$  and  $\operatorname{Re} k_1 > 1$ , we obtain the expansion (1.19) by the set (1.21).

Thus, in classical analysis, we encounter expansions in fractional powers and with constant coefficients, but here we obtain expansions in rather arbitrary complex powers of the independent variable with coefficients that are polynomials in logarithms of this variable. However, there are possible even more complicated expansions of solutions.

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Types of expansions	5		

As  $x \to 0$ , consider asymptotic expansions of solutions to the equation (1.2) of the form

$$y = c_r x^r + \sum_s c_s x^s, \qquad (1.22)$$

where power exponents r and s are complex numbers without points of accumulation,  $\operatorname{Re} s \ge \operatorname{Re} r$ ,  $\operatorname{Re} s$  increase. We define four types of expansions (1.22); the first three of which have finite number of power exponents s with the same real part  $\operatorname{Re} s$  and  $\operatorname{Re} s > \operatorname{Re} r$  (Fig. 3).



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**Type 1.**  $c_r$  and  $c_s$  are constant (*power expansions*);

**Type 2.**  $c_r$  is constant,  $c_s$  are polynomials in log x (*power-logarithmic expansions*);

**Type 3.**  $c_r$  and  $c_s$  are power series in decreasing powers of  $\log x$  (*complicated expansions*).

**Type 4.** There are infinitely many power exponents *s* with a fixed Re *s*, and the convex hull of the points *r* and *s* from (1.22) in the complex plane lies in the angle with the vertex at the point *r*, one of the limiting rays of the angle parallel to the imaginary axis, and the span of the angle being less than  $\pi$  (exotic expansions) (Figs. 4 and 5).

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In addition, we assume that the argument of the complex variable x is limited at one side. The types of asymptotic expansions as  $x \rightarrow \text{const}$  and as  $x \rightarrow \infty$  are defined in a similar way. Similar technique is used for equations having small or big parameters. The power exponents of these parameters are accounted for in the same way as power exponents of variables tending to zero or infinity. Such parameter  $\varepsilon$  can be considered as a dependent variable, satisfying the equation  $\varepsilon' = 0$ . Plane PG. Theory Plane PG. Applications Space PG Other apps of PG

#### The sixth Painlevé equation [3]

#### It has the form

$$y'' = \frac{(y')^2}{2} \left( \frac{1}{y} + \frac{1}{y-1} + \frac{1}{y-x} \right) - y' \left( \frac{1}{x} + \frac{1}{x-1} + \frac{1}{y-x} \right) + \frac{y(y-1)(y-x)}{x^2(x-1)^2} \left[ a + b\frac{x}{y^2} + c\frac{x-1}{(y-1)^2} + d\frac{x(x-1)}{(y-x)^2} \right], \quad (2.1)$$

where *a*, *b*, *c*, *d* are complex parameters, *x* and *y* are complex variables, y' = dy/dx. The equation (2.1) has three singular points x = 0, x = 1, and  $x = \infty$ . After multiplying by common denominator, we obtain the equation as a differential sum. Its support and its polygon, in the case  $a \neq 0$ ,  $b \neq 0$ , are shown in Fig. 6.



Fig. 6

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We found all asymptotic expansions of solutions to the equation (2.1). They comprise 117 families. Among them, there are expansions of all four types. In particular, for a = 1 and c = 0, there is an expansion of the fourth type of the form

$$y = -\frac{1}{\cos [\log (C_1 x)]} + \sum_{\text{Re} s \ge 1} c_s x^s, \qquad (2.2)$$

where  $C_1$  is an arbitrary constant, the coefficients  $c_s$  are constant and uniquely determined. The support of the expansion (2.2) is shown in Fig. 5, where r = i. For  $C_1 = 1$  and real x > 0, the solution (2.2) has infinitely many poles accumulating at the point x = 0.

$$\frac{1}{2\cos(\log x)} = \frac{1}{x^{-i} + x^{i}} = x^{i} \sum_{k=0}^{\infty} \left(-x^{2i}\right)^{k} = x^{-i} \sum_{k=0}^{\infty} \left(-x^{-2i}\right)^{k}.$$

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The Beletsky equat	ion (1956) [4]		

$$(1 + e \cos \nu)\delta'' - 2e \sin \nu\delta' + \mu \sin \delta = 4e \sin \nu$$
 (2.3)

describes plane motions of a satellite around its mass center which is moving along an elliptic orbit with an eccentricity  $e = \text{const} \in [0, 1]$ . In the equation,  $\nu$  is the independent and  $\delta$  is dependent variables, inertial parameter  $\mu = \text{const} \in [-3, 3]$ . The equation (2.3) is singular at e = 1,  $\nu = \pi$ , since the coefficient at the higher derivative vanishes at this point. We introduce local coordinates  $x = \nu - \pi$  and  $\varepsilon = 1 - e$  at the singularity. Then the equation (2.3) takes the form

$$\left[\varepsilon + \frac{1}{2}x^2 + o(x^2, \varepsilon)\right] \frac{d^2\delta}{dx^2} + 2\left[x + o(x, \varepsilon)\right] \frac{d\delta}{dx} + \mu \sin \delta = -4[x + o(x, \varepsilon)].$$
(2.4)

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The support and the polygon of this equation for small coordinates  $x, \varepsilon$  is shown in Fig. 7. The boundary of the polygon  $\Gamma$  consists of three edges and two vertices. The unit vector along the edge  $\Gamma_1^{(1)}$  is (1,0), which corresponds to the variable x. The unit vector along the edge  $\Gamma_2(1)$  is (1, -1/2), which corresponds to the variable  $x/\sqrt{\varepsilon}$ . Using a variable with this type of behavior, we can regularize the equation (2.3) at the singularity and compute its solutions as relaxation oscillations.



Fig. 7

In 1997, we studied the limit equations corresponding to the vertex  $\Gamma_1^{(0)}$  and to the edges  $\Gamma_1^{(1)},\,\Gamma_2^{(1)}.$  Using their solutions, the limits of solutions to the equation (2.3) are matched as  $e \rightarrow 1$ . We found that for e = 1, the limit families of  $2\pi$ -periodic solutions form a complicated structure: the family of symmetric solutions is twisted into the spiral with infinite number of revolutions around the solution  $C = \{\delta = -\nu, \mu = -2\}$  (Fig. 8, schematically), and each convolution of the spiral corresponds to its own family of asymmetric  $2\pi$ -periodic solutions having 4 spirals (2002) (Fig. 9, schematically). Apparently, the solution C is an accumulating point of infinitely many families of  $2\pi$ -periodic solutions and of infinitely many their spirals.

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#### Theory

Let  $X \in \mathbb{C}^m$  be independent and  $Y \in \mathbb{C}^n$  be dependent variables. Suppose  $Z = (X, Y) \in \mathbb{C}^{m+n}$ . A differential monomial a(Z) is the product of an ordinary monomial  $cZ^R = cz_1^{r_1} \dots z_{m+n}^{r_{m+n}}$ , where  $c = \text{const} \in \mathbb{C}$ ,  $R = (r_1, \dots, r_{m+n}) \in \mathbb{R}^{m+n}$ , and a finite number of derivatives of the form

$$\frac{\partial^l y_j}{\partial x_1^{l_1} \dots \partial x_m^{l_m}} \stackrel{\text{def}}{=} \frac{\partial^l y_j}{\partial X^L}, \quad l_j \ge 0, \quad \sum_{j=1}^m l_j = l, \quad L = (l_1, \dots, l_m).$$

A differential monomial a(X) corresponds to its vector *power* exponent  $Q(a) \in \mathbb{R}^{m+n}$  formed by the following rules

$$Q(cZ^R) = R, \ Q(\partial^l y_j / \partial X^L) = (-L, E_j),$$

where  $E_j$  is unit vector. A product of monomials  $a \cdot b$  corresponds to the sum of their vector power exponents:

$$Q(ab) = Q(a) + Q(b).$$

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A differential sum is a sum of differential monomials

$$f(Z) = \sum a_k(Z).$$

A set  $\mathbf{S}(f)$  of vector power exponents  $Q(a_k)$  is called the *support* of the sum f(Z). The closure of the convex hull  $\Gamma(f)$  of the support  $\mathbf{S}(f)$  is called the *polyhedron of the sum* f(Z). Consider a system of equations

$$f_i(X, Y) = 0, \ i = 1, \dots, n,$$
 (4.1)

where  $f_i$  are differential sums. Each equation  $f_i = 0$  corresponds to: its support  $\mathbf{S}(f_i)$ , its polyhedron  $\Gamma(f_i)$  with the set of faces  $\Gamma_{ij}^{(d_i)}$  in the main space  $\mathbb{R}^{m+n}$ , the set of their normal cones  $\mathbf{U}_{ij}^{(d_i)}$  in the dual space  $\mathbb{R}^{m+n}_*$ , and the set of truncated equations  $\hat{f}_{ij}^{(d_i)}(X, Y) = 0$ .

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The set of truncated equations

$$\hat{f}_{ij_i}^{(d_i)}(X,Y) = 0, \ i = 1, \dots, n$$
 (4.2)

is the truncated system if the intersection

$$\mathsf{U}_{1j_1}^{(d_1)} \cap \ldots \cap \mathsf{U}_{nj_n}^{(d_n)} \tag{4.3}$$

is not empty. A solution

$$y_i = \varphi_i(X), i = 1, \ldots, n$$

to the system (4.1) is associated to its normal cone  $\mathbf{u} \subset \mathbb{R}^{m+n}$ . If the normal cone  $\mathbf{u}$  intersects with the cone (4.3), then the asymptotic form  $y_i = \hat{\varphi}_i(X)$ , i = 1, ..., n of this solution satisfies the truncated system (4.2), which is quasihomogeneous.

$$\begin{aligned} &Ap' + (C - B)qr = Mg(y_0\gamma_3 - z_0\gamma_2), \ \gamma'_1 = r\gamma_2 - q\gamma_3, \\ &Bq' + (A - C)pr = Mg(z_0\gamma_1 - x_0\gamma_3), \ \gamma'_2 = p\gamma_3 - r\gamma_1, \\ &Cr' + (B - A)pq = Mg(x_0\gamma_2 - y_0\gamma_1), \ \gamma'_3 = q\gamma_1 - p\gamma_2, \end{aligned}$$

where ' = d/dt, describes the motion of a rigid body with a fixed point. In (4.4),  $A, B, C, x_0, y_0, z_0$ , and Mg are real constants. The system (3.4) has three general first integrals. In the case  $B \neq C$ ,  $x_0 \neq 0, y_0 = z_0 = 0$  N. Kowalewski (1908) reduced the system (4.4) to the system of two equations

$$f_{1} \stackrel{\text{def}}{=} \ddot{\sigma}\tau + \dot{\sigma}\dot{\tau}/2 + a_{1} + a_{2}\sigma + a_{3}\dot{\tau}p + a_{4}\tau + a_{5}p^{2} = 0,$$

$$f_{2} \stackrel{\text{def}}{=} \sigma\ddot{\tau} + \dot{\sigma}\dot{\tau}/2 + b_{1} + b_{2}\dot{\sigma}p + b_{3}\sigma + b_{4}\tau + b_{5}p^{2} = 0,$$
(4.5)

where the dot means differentiation with respect to the new independent variable p,  $\sigma$  and  $\tau$  are new dependent variables,  $a_i, b_i = \text{const.}$ 

This system has two general first integrals. Generically, the supports  $\mathbf{S}(f_i)$  and polyhedrons  $\Gamma(f_i)$  of both equations (4.5) coincide; they are shown in Fig. 10.



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#### The Euler-Poisson equations

We found all power-logarithmic expansions of solutions to the system (4.5) as  $p \rightarrow 0$  and as  $p \rightarrow \infty$  (they comprise 24 families) and 4 families of complicated expansions of solutions. This system does not have expansions of the 4-th type. Using power expansions we obtained all exact solutions of the form of finite sums of real powers of the variable p with complex coefficients. They comprise 12 families. Among them, 7 families were known. All new families are complex.

In the case

$$A = B$$
,  $Mgx_0/B = 1$ ,  $y_0 = z_0 = 0$ ,  $C/B = c$ 

the system (4.4) has a unique parameter  $c \in (0, 2]$ . The system (4.4) has 4 two-parameter families of stationary solutions. On each of these families there are sets  $D_j$  of real stationary solutions near which the system (4.4) is locally integrable as well as the sets  $R_j$  of stationary solutions near which the system (4.4) is locally nonintegrable.

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#### The Euler-Poisson equations

In Fig. 11, there shown the sets  $D_1$ ,  $D_2$ ,  $D_3$  and the curves  $R_1-R_4$  for one of these four families with x = 1/c and  $y = p^0 \gamma_1^0$ , where  $(p, q, r, \gamma_1, \gamma_2, \gamma_3) = (p^0, 0, 0, \pm 1, 0, 0)$  is a stationary solution.



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Other applications of	of Power Geometry		

- Asymptotic forms of solutions to Painlevé equations
- Periodic solutions of the restricted three-body problem
- Analysis of the local integrability of ODE
- Boundary layer on a needle
- Second the second term of the turbulent flow

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