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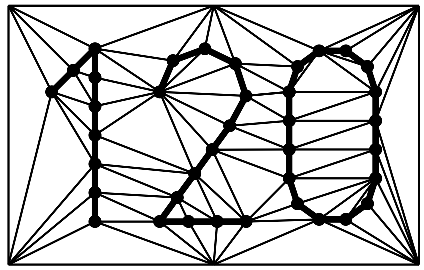

Geometry, Topology, Algebra and Number Theory, Applications

The International Conference dedicated to the 120th anniversary of Boris Nikolaevich Delone (1890-1980)

Abstracts

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Abstracts

Section "Geometry"

Section “Algebra and Number Theory”

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Plenary talks

A survey on spherical designs and Euclidean designs

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Our aim is to study good subsets consisting of finitely many points of the sphere and/or of Euclidean space.

Sometimes we consider not only finite subsets but also finite subsets with weight functions, i.e., cubature formulas in analysis. In this talk, we start with the definition of spherical t-design due to Delsarte-Goethals-Seidel (1977). Then we study spherical t-designs from the viewpoint of algebraic combinatorics. Here, association schemes play an important role. There are natural lower bounds for the size of spherical t-designs, and those which attain one of such lower bounds are called tight spherical t-designs. We discuss the known examples of tight spherical t-designs, and survey the current status of the classification of tight spherical t-designs.

Another main purpose of this talk is to discuss the concept of Euclidean t-designs which are two step generalization of spherical t-designs. Natural lower bounds for the size of Euclidean t-designs, as well as the concept of tight Euclidean t-designs will be discussed. We review the examples and the current status of the classification problem of tight Euclidean t-designs. Some highlights will include our recent complete classification of tight Euclidean 9-designs on two concentric spheres (due to Etsuko Bannai and myself), as well as the new discovery of a tight 6-design on two concentric spheres (due to Etsuko Bannai, Junichi Shigezumi and myself). We discuss the connection between Euclidean designs and the theory of cubature formulas in analysis, and also discuss the role of coherent configurations (generalization of association schemes) in the study of Euclidean t-designs.

Extremal Problems for Convex Lattice Polytopes

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Abstract. In this survey talk I will present several extremal problems, and some solutions, concerning convex lattice polytopes. A polytope is called a lattice polytope if all of its vertices belong to the integer lattice $\mathbb{Z}^d$. Let $\mathcal{P}(n,d)$ denote the family of all convex lattice polytopes, of positive volume, in $\mathbb{R}^d$ with n vertices. The following extremal problems will be considered.

1. minimal volume for $P \in \mathcal{P}(n,d)$,
2. minimal surface area for $P \in \mathcal{P}(n,d)$,
3. minimal lattice width for $P \in \mathcal{P}(n,d)$,
4. maximal n such that a (large) convex set $K \subset \mathbb{R}^d$ contains and element of $\mathcal{P}(n,d)$, in other words, the maximal number of lattice points in K that are in convex position.

These problems are related to a question of V I Arnold from 1980 asking for the number of (equivalence classes of) lattice polytopes of volume (at most) V in d-dimensional space. Here two convex lattice polytopes are equivalent if one can be carried to the other by a lattice preserving affine transformation.

Boundary rigidity, volume minimality, and minimal surfaces in L_∞: a survey[1]

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A Riemannian manifold with boundary is said to be boundary rigid if its metric is uniquely determined by the boundary distance function, that is the restriction of the distance function to the boundary. Loosely speaking, this means that the Riemannian metric can be recovered from measuring distances between boundary points only. The goal is to show that certain classes of metrics are boundary rigid (and, ideally, to suggest a procedure for recovering the metric).

To visualize that, imagine that one wants to find out what the Earth is made of. More generally, one wants to find out what is inside a solid body made of different materials (in other words, properties of the medium change from point to point). The speed of sound depends on the material. One can "tap"at some points of the surface of the body and "listen when the sound gets to other points". The question is if this information is enough to determine what is inside.

This problem has been extensively studied from PDE viewpoint: the distance between boundary points can be interpreted as a "travel time"for a solution of the wave equation. Hence this becomes a classic Inverse Problem when we have some information about solutions of a certain PDE and want to recover its coefficients. For instance such problems naturally arise in geophysics (when we want to find out what is inside the Earth by sending sound waves), medical imaging etc.

In a joint project with S. Ivanov we suggest an alternative geometric approach to this problem. In our earlier work, using this approach we were able to show boundary rigidity for metrics close to flat ones (in all dimensions), thus giving the first class of boundary rigid metrics of nonЏconstant curvature beyond two dimensions. We were now able to extend this result to include metrics close to a hyperbolic one.

The approach is grew up from another long-term project of studying surface area functionals in normed spaces, which we have been working on it for more than ten years. There are a number of related issues regarding area-minimizing surfaces in Riemannian manifold. The talk gives a non-technical survey of ideas involved. It assumes no background in inverse problems and is supposed to be accessible to a general math audience (in other words, we will not get into any technical details of the proofs).

[1] on a joint work with S. Ivanov.

On the polyhedral product functor: a method of decomposition for moment-angle complexes

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Spaces which are now called *(generalized) moment-angle complexes* or values of the *"polyhedral product functor"* have been studied by topologists since the 1960's thesis of G. Porter. In the 1970's E. B. Vinberg developed some of their features. In the late 1980's S. Lopez de Medrano developed beautiful properties of intersections of quadrics with recent further developments in joint work with S. Gitler.

In seminal work during the early 1990's, M. Davis and T. Januszkiewicz introduced manifolds now often called quasi-toric manifolds. They showed that every quasi-toric manifold is the quotient of a moment-angle complex by the free action of a real torus. The moment-angle complex is denoted $Z(K;(D^2,S^1))$ where K is a finite simplicial complex.

The integral cohomology of the spaces $Z(K;(D^2,S^1))$ has been studied by Goresky-MacPherson, Buchstaber-Panov, Panov, Baskakov, Hochster, and Franz. Among others who have worked extensively on generalized moment-angle complexes are Notbohm-Ray, Grbic-Theriault, Strickland and Kamiyama-Tsukuda. Buchstaber-Panov synthesized several different developments in this subject. The direction of this lecture is guided by work of Denham-Suciu.

This lecture is a survey of recent work on generalized moment-angle complexes as well as related spaces. One of the results given here is a natural decomposition for the suspension of the generalized moment-angle complex, the value of the suspension of the "polyhedral product functor".

Since the decomposition is geometric, an analogous homological decomposition for a generalized moment-angle complex applies for any homology theory. This last decomposition specializes to the homological decompositions in the work of several authors cited above. Furthermore, this decomposition gives an additive decomposition for the Stanley-Reisner ring of a finite simplicial complex extended to other natural settings. Applications to the real K-theory of moment-angle complexes as well as associated cup-product structures are given. Applications to robotic motion are illustrated via video clips.

This lecture is based on joint work with A. Bahri, M. Bendersky, and S. Gitler. The application to robotics is based on joint work with D. Koditschek and G. C. Lynch.

Sets of links of vertices of triangulated manifolds and combinatorial approach to Steenrod's problem on realisation of cycles

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To each triangulated manifold one can assign the set of links of its vertices. The link of a vertex describes the local combinatorial structure of the triangulation in a neighbourhood of the vertex. Thus the set of links of vertices of a triangulation can be interpreted as the set of local combinatorial data characterizing the triangulation. We consider a compatibility problem for such local combinatorial data. This problem can be formulated in the following way. For a given set of combinatorial spheres, does there exist a triangulated manifold with such set of links of vertices? We are mostly interested in an oriented version of this problem. Our aim is to obtain a non-trivial sufficient condition for compatibility of a set of links of vertices. We shall describe an explicit construction that, under certain natural conditions, allows us to realise a multiple of a given set of oriented combinatorial spheres as the set of links of vertices of a combinatorial manifold.

Further, we are going to discuss an application of this construction to N. Steenrod's problem on realisation of cycles. It is well known that according to a result of R. Thom, any n-dimensional integral homology class z of any topological space X can be realised with some multiplicity by an image of an oriented smooth closed manifold N^n. Our new approach is based on an explicit combinatorial procedure for resolving singularities of a cycle. We give an explicit combinatorial construction that, for a given homology class z, yields a manifold N^n and its mapping to X which realises the class z with some multiplicity. Moreover, the obtained manifold N^n appears to be a finite-fold non-ramified covering over a very interesting special manifold M^n, which can be regarded either as an isospectral manifold of symmetric tridiagonal real $(n+1)\times(n+1)$-matrices or as a small covering over a permutohedron.

[1]The work is partially supproted by the Grant of the President of Russian Federation for Support of Young Russian Scientists (grant 4220.2009.1) and the Russian Foundation for Basic Research (grant 08-01-00541).

Packing and Covering of Convex Bodies

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1 Introduction

Let C be an o-symmetric convex body and L a lattice in $\mathbb{E}^d$. Then $\delta(C)$ and $\vartheta(C)$ denote the maximum lattice packing, resp. the minimum lattice covering density of C. Let $\varrho \geq 0$ be maximum and $\sigma \geq 0$ minimum such that L provides a packing of ϱC and a covering of σC. Then $\delta(C, L)$ and $\vartheta(C, L)$ are the densities of this packing resp. this covering. Let $k(C, L)$ be the kissing number of the lattice packing of ϱC provided by L and $\underline{k}(C), \overline{k}(C)$ the minimum, resp. maximum kissing number of a lattice packing of C of maximum density.

Over the last hundred years important contributions to the following problems have been given:

(i) Upper and lower estimates for $\delta(C), \vartheta(C)$ and $\delta(B^d), \vartheta(B^d)$ (Minkowski-Hlawka, Rogers).

(ii) Upper and lower estimates for $k(C, L), \underline{k}(C), \overline{k}(C)$ (Minkowski, Swinnerton-Dyer).

(iii) Criteria for local maxima of $\delta(B^d, L)$ and local minima of $\vartheta(B^d, L)$ (Voronoi, Barnes and Dickson, Delone et al, Schürmann and Vallentin).

Concerning the following problems, not much progress has been achieved.

(iv) Uniqueness of densest lattice packings and thinnest lattice coverings.

(v) Determination of densest lattice packings and thinnest lattice coverings, algorithms. (Exception: Betke–Henk)

In this lecture we make some remarks on problems (iv), (iii) and (ii).

2 Uniqueness of Densest Lattice Packing and Thinnest Lattice Covering

In the rare cases when the densest lattice packings or the thinnest lattice coverings of a convex body are known, these results may, sometimes, be interpreted of (possibly weak) uniqueness results. In the case of B^d there are, up to rotation, only finitely many densest lattice packings. This is a consequence of a theorem of Voronoi. Analogous statement holds for coverings as shown by Schürmann and Vallentin. Except for these cases we are not aware of any pertinent uniqueness theorem.

Using Baire categories, one can prove the following results

Theorem 1. There is a constant $a(d) \geq 1$, such that for a generic o-symmetric convex body C in $\mathbb{E}^d$ there are at most $a(d)$ lattice packings of maximum density. For $d = 2, 3$ one may put $a(d) = 1$.

Theorem 2. There is a constant $b(d) \geq 1$, such that for a generic o-symmetric convex body C in $\mathbb{E}^d$ there are at most $b(d)$ lattice coverings of minimum density. For $d = 2$ one may put $b(d) = 1$.

Problem 1. Show that $a(d) = b(d) = 1$ for all d,

i.e., the densest lattice packings and the thinnest lattice coverings of a generic љo-symmetric convex body are unique in all dimensions.

3 Lattice Packings and Coverings of Extreme Density

A classical criterion of Voronoi says that a positive definite quadratic form on $\mathbb{E}^d$ is extreme (i.e. locally maximum among all neighbouring form) if and only if it is perfect and eutactic. Equivalently, a lattice packing of B^d has locally maximum density if and only if it is perfect and eutactic.

Using the following refined notions of maximality of (local, upper)

semi-stationarity,
stationarity,
maximality, and
ultra-maximality

and the Voronoi type notions of

semi-eutaxy,
eutaxy, and
perfection

the following results hold:

Theorem 3. The following statements on B^d, L, δ hold:

(i) L is semi-stationary if and only if it is semi-eutactic.
(ii) No lattice is stationary.
(iii) L is maximum if and only if it is perfect and eutactic.
(iv) L is ultra-maximum if and only if it is perfect and eutactic.

Statement (iii) is Voronoi's criterion. The unexpected consequence is that each lattice packing of maximum density has already ultra-maximum density.

Using suitable generalizations of the extremum and Voronoi type notions stated earlier, one can prove the following partial extension of Theorem 3, where C is a smooth and strictly convex o-symmetric convex body.

Theorem 4. The following statements on C, L, δ hold:

(i) L is semi-stationary if and only if it is semi-eutactic.
(ii) No lattice is stationary.
(iii) L is ultra-maximum if and only if it is perfect and eutactic.

The problem to characterize maximum lattices remains open.

Problem 2. Show that in sufficiently high dimensions for a generic o-symmetric convex body C the densest lattice packing is unique and ultra-maximum.

For coverings we could prove only the following criterion, where we used the notions of (local, lower) semi-stationarity,

stationarity, and
ultra-minimality

and the Voronoi type notions of

para-completeness,
completeness, and
ultra-completeness.

Theorem 5. For B^d, L, ϑ the following statements hold:

(i) L is semi-stationary if and only if it is para-complete.
(ii) L is stationary if and only if it is complete.
(iii) L is ultra-minimum if and only if it is ultra-complete.

The problem to characterize minimum lattices remains open.

Problem 3. Extend Theorem 5 to o-symmetric convex bodies and show that for a generic o-symmetric convex body the thinnest lattice covering is unique and ultra-minimum.

4 Kissing Number of Generic Convex Bodies

A result of Swinnerton-Dyer implies that for each o-symmetric convex body C holds

$$\underline{k}(C) \geq d(d+1),$$

while the author has shown that in the generic case holds

$$\overline{k}(C) \leq 2d^2.$$

(In the general case $\overline{k}(C) \leq 3^d - 1$ by an old estimate of Minkowski.)

Problem 4. Show that in the generic case the densest lattice packing is unique and

$$\underline{k}(C) = \overline{k}(C) = 2d^2.$$

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The Quasi-triangulation and The Beta-complex: Theory and Applications

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The Voronoi/Delaunay structures are everywhere in nature and useful for understanding the spatial structure of a point set. Being powerful computational tools, their generalization has been made in various directions including the Voronoi diagram of spherical balls. The Voronoi diagram of spherical balls nicely defines the proximity among the balls where the Euclidean distance is used from the spherical boundary of balls. Like its counterparts of the ordinary Voronoi diagram of points or the power diagram, the dual structure can be more convenient in both representing and traversing the topology structure of the Voronoi diagram. However, unlike the Delaunay triangulation and the regular triangulation, the dual structure of the Voronoi diagram of balls, the quasi-triangulation, is not a simplicial complex and creates a number of anomaly cases which cause difficulties in the representation and traversal of topology.

This talk will introduce the Voronoi diagram of balls and its quasi-triangulation, particularly in the three-dimensional space. Given its definition, the properties of the quasi-triangulation, including the anomalies, will be presented with the underlying data structure to store its topology. Based on the quasi-triangulation, we define a new geometric structure called the beta-complex which concisely yet efficiently represents the proximity among all spherical balls within the boundary of the input ball set, where its boundary is appropriately defined. It turns out that thus defined the beta-complex can be used to efficiently solve geometry and topology problems for the ball set. Among many potential application areas, the structural molecular biology is the utmost application area because the beta-complex immediately and efficiently solves many geometry problems related to important structural molecular biology problems: Examples include the computation of the molecular surface, the extraction of pockets on the boundary of molecule, the computation of areas of various types of surfaces defined on a molecule, the computation of various kinds of volumes defined on a molecule, the docking simulation, etc. We will also demonstrate our molecular modeling and analysis software, BetaMol, which is entirely based on the unified, single representation of the quasi-triangulation and the beta-complex.

Cohomological rigidity problems in toric topology

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Cohomological rigidity problems in toric topology

Classification of compact smooth toric varieties (which we call toric manifolds) as varieties reduces to classification of their fans as is well-known. However, not much is known for classification of toric manifolds as smooth manifolds. One interesting and naive question is

Cohomological rigidity problem for toric manifolds ([3]). Are two toric manifolds diffeomorphic (or homeomorphic) if their cohomology rings are isomorphic as graded rings?

Some partial affirmative solution and no counterexample is known to the problem so far. Similar questions can be asked for polytopes ([1]), real toric manifolds ([2]) and symplectic toric manifolds ([4]). In this talk I will discuss these problems.

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What can we do with Diophantine problems and what we cannot do

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In the talk I'll survey numerous theorems (obtained mainly by logicians and computer scientists) about impossibility of algorithms for diverse Diophantine problems. However, such "negative" results are often obtained as corollaries of "positive" theorems about possibilities to construct Diophantine problems with special properties. Below are just two examples.

1. We can find a particular polynomial $P(a, x_1, \dots, x_n)$ with integer coefficients such that

- for every value of the parameter a the equation

$$P(a, x_1, \dots, x_n) = y + 4^y \tag{$*$}$$

  has at most one solution;

- for every function $\beta(a)$ defined and effectively computable for all values of a there is a number a_β such that for $a = a_\beta$ the equation $(*)$ has a solution $x_1 = x_{1,\beta}, \dots, x_n = x_{n,\beta}, y = y_\beta$ and this (unique) solution satisfies the inequality

$$\max\{x_{1,\beta}, \dots, x_{n,\beta}, y_\beta\} > \beta(a_\beta).$$

In other words, $(*)$ is a "principally uneffectivizable" equation, that is, we can bound the number of its solutions (by 1 for every value of the parameter) but cannot bound solutions themselves by any computable function of the parameter.

Open problem. Could we replace $(*)$ by a suitable genuine Diophantine equation, that is, without exponentiation?

2. We can find natural numbers $\mathbf{d}$ and $\mathbf{n}$ such that for every polynomial $P(a_1, \dots, a_m, x_1, \dots, x_n)$ with integer coefficients (having any degree and arbitrary number of variables) we can effectively construct another polynomial $Q(a_1, \dots, a_m, y_1, \dots, y_{\mathbf{n}})$ with integer coefficient with $m + \mathbf{n}$ variables having degree $\mathbf{d}$ with respect to variables $y_1, \dots, y_{\mathbf{n}}$ and such that both Diophantine equations

$$P(a_1, \dots, a_m, x_1, \dots, x_n) = 0 \tag{$**$}$$

and

$$Q(a_1, \dots, a_m, y_1, \dots, y_{\mathbf{n}}) = 0 \tag{$***$}$$

are solvable (in $x_1, \dots, x_n$ and $y_1, \dots, y_{\mathbf{n}}$ respectively) for the same values of the parameters $a_1, \dots, a_m$.

In other words, the traditional classifications of difficulties of Diophantine equations ("equations of degree 1", "equations of degree 2", ..., and "equations in one unknown", "equations in two unknowns", ...) collapse from some point on.

Today we know that for such universal bound $\langle \mathbf{d}, \mathbf{n} \rangle$ we can take any of the pairs

$$\langle 4, 58\rangle,\ \langle 8, 38\rangle,\ \langle 12, 32\rangle,\ \langle 16, 29\rangle, \langle 20, 28\rangle,\ \langle 24, 26\rangle,\ \langle 28, 25\rangle,$$
$$\langle 36, 24\rangle,\ \langle 96, 21\rangle,\ \langle 2668, 19\rangle,\ \langle 2\times 10^5, 14\rangle, \langle 6.6\times 10^{43},\ 13\rangle,$$
$$\langle 1.3\times 10^{44}, 12\rangle,\ \langle 4.6\times 10^{44}, 11\rangle,\ \langle 8.6\times 10^{44}, 10\rangle,\ \langle 1.6\times 10^{45}, 9\rangle$$

in the case when the unknowns range over natural numbers; in the case when the unknowns range over integers the values are a bit bigger. If we allow (iterated) usage of exponential function 2^z in construction of equation $(***)$, then the number of unknowns can be as small as $\mathbf{n} = 3$.

Open problem. Could we find similar universal bound $\langle \tilde{\mathbf{d}}, \mathbf{n} \rangle$ where $\tilde{\mathbf{d}}$ would be the total degree of the polynomial $Q(a_1, \dots, a_m, y_1, \dots, y_{\mathbf{n}})$, that is, with respect to all the variables $a_1, \dots, a_m, y_1, \dots, y_{\mathbf{n}}$?

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Effective results in diophantine equations

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We plan to give a survey of effective results and methods concerning estimates for the number of solutions of diophantine equations as well as results concerning bounds for the solutions.

1. Discussion of Delone's results about diophantine equations.

2. Approximation of algebraic numbers by rationals (from A. Thue to K. Roth).

3. Bounds for linear forms in algebraic numbers and Subspace Theorem (W. Schmidt).

4. Linear forms in logarithms of algebraic numbers (A. Gelfond, A. Baker and followers). Eнective bounds for solutions of diophantine equations.

5. Catalan's problem.

6. Algorithms and computers.

Number-theoretical properties of hyperelliptic fields.

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Resonance varieties of arrangement complements, Milnor fiber, and Bernstein polynomials

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In the talk we recall the definition of the resonance varieties and their main properties for hyperplane complements. Then we discuss their most recent applications to cohomology of Milnor fiber and roots of Bernstein polynomials for products of linear forms.

Section “Geometry”

The extremal spheres theorems[1]

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Consider a polygon P and all neighboring circles (circles going through three consecutive vertices of P). We say that a neighboring circle is extremal if it is empty (no vertices of P inside) or full (no vertices of P outside). It is well known that for any convex polygon there exist at least two empty and at least two full circles, i.e. at least four extremal circles. In 1990 Schatteman considered a generalization of this theorem for convex polytopes in d-dimensional Euclidean space. Namely, he claimed that there exist at least $2d$ extremal neighboring spheres.

We show that there are certain gaps in Schatteman's proof. His proof is based on the Bruggesser-Mani shelling method. We show that using this method it is possible to prove that there are at least $d+1$ extremal neighboring spheres. However, the existence problem of $2d$ extremal neighboring spheres is still open.

[1] This is a joint work with Alexey Glazyrin, Oleg Musin, Alexey Tarasov.

Algebra versus analysis in the theory of flexible polyhedra

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A polyhedron (more precisely, a polyhedral surface) is said to be *flexible* if its spatial shape can be changed continuously due to changes of its dihedral angles only, i. e., if every face remains self-congruent during the flex. In other words, a polyhedron P_0 is flexible if it is included in a continuous family $\{P_t\}$, $0 \leqslant t \leqslant 1$, of polyhedra P_t such that, for every t, the corresponding faces of P_0 and P_t are congruent while polyhedra P_0 and P_t are not congruent. In general, self-intersections are possible both for P_0 and P_t. Without loss of generality we assume that the faces of the polyhedra are triangular.

Flexible self-intersection free sphere-homeomorphic polyhedra in Euclidean 3-space were constructed by R. Connelly [3]. During last 30 years, many non-trivial properties of flexible polyhedra were discovered. We formulate two of them in a form that is convenient for our purposes.

Let P be a closed oriented polyhedron in $\mathbb{R}^3$, let E be the set of its edges, let $|\ell|$ be the length of edge ℓ, and let $\alpha(\ell)$ be the dihedral angle of P at edge ℓ measured from inside of P. The sum

$$M(P) = \frac{1}{2}\sum_{\ell \in E} |\ell|\big(\pi - \alpha(\ell)\big)$$

is called the *total mean curvature* of P.

Theorem 1. *Let P_0 be a flexible polyhedron in $\mathbb{R}^3$ and let $\{P_t\}$, $0 \leqslant t \leqslant 1$, be its flex. The total mean curvature $M(P_t)$ is independent of t.*

Theorem 1 was obtained by R. Alexander [1] as an obvious corollary of Theorem 2, while the latter was proved in [1] with the help of the Stokes theorem, i. e., by means of Analysis.

Theorem 2. *Let P be a closed oriented polyhedron in $\mathbb{R}^3$, let $\boldsymbol{w}$ be its infinitesimal flex, and let $P(t) = \{\boldsymbol{r} + t\boldsymbol{w} | \boldsymbol{r} \in P\}$. Then $\frac{d}{dt}\big|_{t=0} M\big(P(t)\big) = 0$.*

Another important property of the flexible polyhedra is given by the following theorem.

Theorem 3. *If $\{P_t\}$ is a flex of an orientable polyhedron in $\mathbb{R}^3$, then the oriented volume of P_t is constant in t.*

Theorem 3 was obtained by I.Kh. Sabitov [4] as an obvious corollary of Theorem 4, while the latter was proved in [4] with the help of the theory of resultants, i. e., by means of Algebra.

Theorem 4. *For the set $\mathcal{P}_K$ of all (not necessarily flexible) closed polyhedra in $\mathbb{R}^3$ with triangular faces and with a prescribed combinatorial structure K there exists a universal polynomial $\mathfrak{p}_K$ of a single variable whose coefficients are universal polynomials in the edge lengths of a polyhedron $P \in \mathcal{P}_K$ and such that the oriented the volume of any $P \in \mathcal{P}_K$ is a root of $\mathfrak{p}_K$.*

We prove that Theorem 1 cannot be proved by means of Algebra and Theorem 3 cannot be proved by means of Analisis. In particular, we prove the followng theorem that may be of indepented interest.

Theorem 5. *The total mean curvature of any closed oriented polyhedron in $\mathbb{R}^3$ is not an algebraic function of its edge lengths.*

Full text of this talk is available in [2].

[1] The author is supported by the Russian Foundation for Basic Research (grant 10–01–91000–ANF) and the Federal Program 'Research and educational resourses of innovative Russia in 2009–2013' (contract 02.740.11.0457).

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On submanifolds of negative curvature in Euclidean spaces

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In the theory of submanifolds with negative curvature there exist some number of interesting and unsolved problems, connected with influence of curvature and codimention on submanifold.

We will exposer results about isometric immersions of the Lobachevsky space into Euclidean spaces, as well we will give one method to construct different isometric immersions with non flat normal connection.

By using the Rozendorn surface, that is an isometric immersions of the Lobachevsky plane into E^5 in form of regular surface F^2, we construct 3-dimensional submanifold F^3 in F^5, contains F^2 and such that the sectional curvature of F^3 for plane, tangent F^2 , is negative and separate from zero.

Метрические свойства ломаных Понселе

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Теоремы о замыкании, или теоремы типа Понселе, широко изучаются в литературе и применяются в классической геометрии, теории алгебра-ических кривых, комплексном анализе, дифференциальных уравнениях (см. библиографию в [2]Ц[4]). Мы ограничимся случаем двух окружностей.

Везде далее α и β - две окружности (β лежит внутри α) радиусов R, r, расстояние между центрами которых равно d. Под ломаной Понселе этих окружностей будем понимать ломаную $v_1 \dots v_n$, у которой все вершины - $A_1, A_2, \dots, A_{n+1}$ - лежат на окружности α, а все звенья - $v_1, v_2, \dots, v_n$ - касаются β. Если ломаная замкнута, т.е. $A_{n+1} = A_1$, и не имеет самопере-сечений, то будем ее называть многоугольником Понселе.

Мы представляем следующие результаты:

1) получено обобщение *(теорема 1)* и новое доказательство (геометри-ческое) теоремы Радича и Калимана [1];

2) получен общий принцип нахождения формул на условия замыкания для двух окружностей *(теорема 2)* и

3) связь таких формул для n и $2n$ *(теорема 3)*;

4) получены условия замыкания для ломаных Понселе, обобщающие известные формулы Эйлера для $n = 3$ и Фусса для $n = 4$, из которых выведены формулы для $n = 6$ и $n = 8$.

Теорема 1. *Пусть $v_1 \dots v_n$ - ломаная Понселе α и β с началом X и концом Y, а звенья v_1 и v_n касаются β в точках X' и Y' соответственно. Тогда для всех таких ломаных $\frac{XY}{XX'+YY'}$ - величина постоянная.*

Обозначим данную величину через $k_n(\alpha, \beta)$. Из большой теоремы Понселе [5] следует, что все такие прямые XY касаются одной окружности γ_n соосной с α и β. Радич и Калиман установили, что $k_2(\alpha, \beta) = \frac{2Rr}{R^2-d^2}$. Отсюда следуют известные формулы Эйлера

$$\frac{1}{R-d} + \frac{1}{R+d} = \frac{1}{r} \quad (n = 3)$$

и Фусса

$$\frac{1}{(R-d)^2} + \frac{1}{(R+d)^2} = \frac{1}{r^2} \quad (n = 4)$$

Далее мы находим общий принцип получения формул на условия замыкания. Если окружности α, β и γ принадлежат одному пучку, то для любой точки окружности α отношение ее степеней относительно окружностей γ и β постояно. Обозначим эту величину через $k_\alpha(\frac{\gamma}{\beta})$. Введем отображения $G_i : \mathbb{R}^3 \to \mathbb{R}^3$, $i \geqslant 2$:

$$G_i(R, r, d) = (R, \sqrt{R^2 + k_i^2 d^2 - k_i(R^2 - r^2 + d^2)}, k_i d),$$

где $k_i = k_\alpha(\frac{\gamma_i}{\beta})$ и

$$F_n(R, r, d) = k_{n-1}(\alpha, \beta) - 1.$$

Тогда $F_n(R, r, d) = 0$ задает условие существования n-угольника Понселе для окружностей α и β.

Теорема 2. *Окружности* α *и* β *обладают* $n-$*свойством Понселе, где* $n = n_1 \dots n_r$, $n_i \in \mathbb{N}$. *Тогда* $F_n = F_{n_{\sigma(1)}} \circ G_{n_{\sigma(2)}} \circ \ldots \circ G_{n_{\sigma(r)}}$, *где* $G_{n_{\sigma(i)}} = G_{n_{\sigma(i)}}(\alpha, \gamma_{t_i})$, $t_i = \prod_{j=i+1}^{r} n_{\sigma(j)}$, $i = 2, \dots, r-1$, $t_r = 1$, $\sigma \in \mathbf{S}_r$.

Следующая теорема устанавливает соотношение между формулами на условие замыкания для n и $2n$.

Теорема 3. Для $\forall n \geqslant 3$ имеем $F_{2n}(R; r; d) = F_n\left(R; \frac{2Rr^2(R^2+d^2)}{(R^2-d^2)^2} - R; \frac{4R^2r^2d}{(R^2-d^2)^2}\right)$.

Из теоремы 3 и формул Эйлера и Фусса получаем формулы для 6-угольника Понселе:

$$\frac{1}{(R^2-d^2)^2 - 4Rr^2d} + \frac{1}{(R^2-d^2)^2 + 4Rr^2d} = \frac{1}{2r^2(R^2+d^2)^2 - (R^2-d^2)^2}$$

и для 8-угольника Понселе:

$$\frac{1}{((R^2-d^2)^2 - 4Rr^2d)^2} + \frac{1}{((R^2-d^2)^2 + 4Rr^2d)^2} = \frac{1}{(2r^2(R^2+d^2)^2 - (R^2-d^2)^2)^2}$$

Мы также исследуем связь данных формул с известными формулами Кэли (формулами, выражающими условия замыкания ломаной Понселе в терминах определителей специальных матриц).

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A survey on spherical designs and Euclidean designs

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The concept of Euclidean t-design was first defined by Neumaier-Seidel (1988), as a two-step generalization of the concept of spherical t-design. Neumaier-Seidel and Delsarte-Seidel gave natural lower bounds of the cardinalities of Euclidean t-designs for even integer t. However the natural lower bounds of the cardinalities of Euclidean t- designs are already given by Möller (1976) in more general context, i.e., in terms of cubature formula.

In this talk we give the definition of the Euclidean t-designs. Then introduce some basic facts on Euclidean t-designs. Give the definition of the tightness of Euclidean t-designs. It is known that tight t-designs on p concentric spheres in $\mathbb{R}^n$ have the structures of coherent configurations if p is not so large. In particular tight t-designs on two concentric spheres have the structures of coherent configurations. We discuss the classification problem of tight t-designs on two concentric spheres in $\mathbb{R}^n$ using this property.

This is joint work with Eiichi Bannai.

Illuminating Ball-Polyhedra

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Let $\mathbf{K}$ be a convex body (i.e. a compact convex set with nonempty interior) in the d-dimensional Euclidean space $\mathbb{E}^d$, $d \geqslant 2$. According to Boltyanski [2] the direction $\mathbf{v} \in \mathbb{S}^{d-1}$ (i.e. the unit vector $\mathbf{v}$ of $\mathbb{E}^d$) illuminates the boundary point $\mathbf{b}$ of $\mathbf{K}$ if the halfline emanating from $\mathbf{b}$ having direction vector $\mathbf{v}$ intersects the interior of $\mathbf{K}$, where $\mathbb{S}^{d-1} \subset \mathbb{E}^d$ denotes the $(d-1)$-dimensional unit sphere centered at the origin $\mathbf{o}$ of $\mathbb{E}^d$. Furthermore, the directions $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ illuminate $\mathbf{K}$ if each boundary point of $\mathbf{K}$ is illuminated by at least one of the directions $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$. Finally, the smallest n for which there exist n directions that illuminate $\mathbf{K}$ is called the *illumination number* of $\mathbf{K}$ denoted by $I(\mathbf{K})$. An equivalent but somewhat different looking concept of illumination was introduced by Hadwiger in [3]. There he proposed to use point sources instead of directions for the illumination of convex bodies. Based on these circumstances the following conjecture, that was independently raised by Boltyanski [2] and Hadwiger [3] in 1960, is called the *Boltyanski–Hadwiger Illumination Conjecture*: The illumination number $I(\mathbf{K})$ of any convex body $\mathbf{K}$ in $\mathbb{E}^d$, is at most 2^d and $I(\mathbf{K}) = 2^d$ if and only if $\mathbf{K}$ is an affine d-cube.

Let $\mathbf{K}$ be a convex body in $\mathbb{E}^d$ and let F be a face of $\mathbf{K}$ (i.e. let F be the intersection of a supporting hyperplane of $\mathbf{K}$ with the boundary of $\mathbf{K}$). Recall that the *Gauss image* $\nu(F)$ of the face F is the set of all points (i.e. unit vectors) $\mathbf{u} \in \mathbb{S}^{d-1} \subset \mathbb{E}^d$ with the property that the supporting hyperplane of $\mathbf{K}$ with outer normal vector $\mathbf{u}$ contains F. It is easy to see that the Gauss images of distinct faces of $\mathbf{K}$ have disjoint relative interiors in $\mathbb{S}^{d-1}$ and $\nu(F)$ is compact and spherically convex for any face F. (Recall that a set $Y \subset \mathbb{S}^{d-1}$ is spherically convex if it is contained in an open hemisphere of $\mathbb{S}^{d-1}$ and for every $\mathbf{y}_1, \mathbf{y}_2 \in Y$ the shorter great-circular arc of $\mathbb{S}^{d-1}$ connecting $\mathbf{y}_1$ with $\mathbf{y}_2$ is in Y.) Now, let $Y \subset \mathbb{S}^{d-1}$ be a set of finitely many points. Then the *covering radius* of Y is the smallest positive real number r with the property that the family of $(d-1)$-dimensional closed spherical balls of (angular) radii r centered at the points of Y cover $\mathbb{S}^{d-1}$. The following, rather basic principle, can be quite useful for estimating the illumination numbers of some convex bodies in particular, in low dimensions.

Theorem 1. *Let $\mathbf{K} \subset \mathbb{E}^d$, $d \geqslant 3$ be a convex body and let r be a positive real number with the property that the Gauss image $\nu(F)$ of any face F of $\mathbf{K}$ can be covered by a $(d-1)$-dimensional closed spherical ball of (angular) radius r in $\mathbb{S}^{d-1}$. Moreover, assume that there exist k points of $\mathbb{S}^{d-1}$ with covering radius R satisfying the inequality $r + R \leq 90^\circ$. Then $I(\mathbf{K}) \leq k$.*

In what follows, sets that we are going to study, will include intersections of finitely many congruent closed d-dimensional balls in $\mathbb{E}^d$. In fact, one may assume that the congruent d-dimensional balls in question are of unit radius that is they are unit balls of $\mathbb{E}^d$. Also, it is natural to assume that removing any of the unit balls defining the intersection in question yields the intersection of the remaining unit balls to become a larger set. The sets obtained in this way are called *ball-polyhedra*. For a comprehensive list of properties of ball-polyhedra we refer the interested reader to the recent paper [1] of the author, Lángi, Naszódi and Papez.

Theorem 1 implies the following statement.

Corollary 1. *Let $\mathbf{B}[X]$ be a ball-polyhedron in $\mathbb{E}^3$, which is the intersection of the closed 3-dimensional unit balls centered at the points of $X \subset \mathbb{E}^3$.*

(i) If the Euclidean diameter $\mathrm{diam}(X)$ *of* X *satisfies* $0 < \mathrm{diam}(X) \leq 0.577$*, then* $I(\mathbf{B}[X]) = 4$*;*

(ii) If the Euclidean diameter $\mathrm{diam}(X)$ *of* X *satisfies* $0.577 < \mathrm{diam}(X) \leq 0.774$*, then* $I(\mathbf{B}[X]) \leq 5$.

In connection with this, it is natural to expect a stronger result to hold namely, that *the illumination number of any ball-polyhedron in* $\mathbb{E}^3$ is always less than 8 (in particular, maybe it *is always at most* 5).

It is clear that estimates similar to Corollary 1 exist in higher dimensions. However, the following approach based on the elegant paper [4] of Schramm is a more efficient one in particular, if the dimension is sufficiently large. More concretely, by taking a closer look of the proof in [4], and making the necessary modifications, it turnes out, that the main result of Schramm [4] on estimating the illumination numbers of convex bodies of constant width can be improved as well as extended to the following family of convex bodies that is much larger than the family of convex bodies of constant width and includes the family of "fat" ball-polyhedra. Thus, we have obtained the following new result.

Theorem 2. *Let* $X \subset \mathbb{E}^d$*,* $d \geq 3$ *be an arbitrary compact set with Euclidean diameter* $\mathrm{diam}(X) \leq 1$ *and let* $\mathbf{B}[X]$ *be the intersection of the closed d-dimensional unit balls centered at the points of* X*. Then*

$$I(\mathbf{B}[X]) < 4\left(\frac{\pi}{3}\right)^{\frac{1}{2}} d^{\frac{3}{2}}(3+\ln d)\left(\frac{3}{2}\right)^{\frac{d}{2}} < 5d^{\frac{3}{2}}(4+\ln d)\left(\frac{3}{2}\right)^{\frac{d}{2}}.$$

This proves the Illumination Conjecture for all "fat" ball-polyhedra of dimension at least 15.

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О фундаментальном многограннике дискретной группы движений гиперболического пространства

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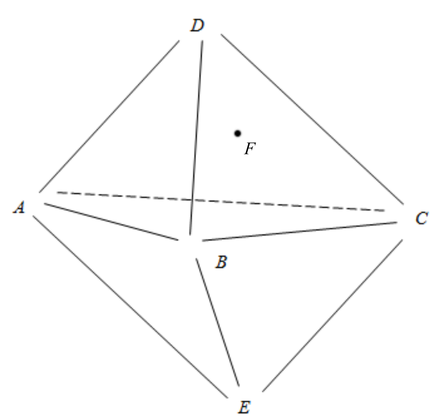
В работе [1] показано, что трехмерное гиперболическое пространство разбивается нормально и правильно выпуклыми многогранниками, двугранные углы которых имеют два вида: рациональными относительно π и иррациональными относительно π .

В данной работе построен фундаментальный многогранник для дискретной группы движений Λ^3, все двугранные углы которого иррациональны относительно π. Более того, множество таких многогранников определяется тремя непрерывными параметрами.

Рассмотрим в гиперболическом пространстве треугольную бипирамиду со всеми бесконечно удаленными вершинами. Обозначим ее вершины буквами A, B, C, D, E, где двугранные углы при ребрах AB, BC и CA равны $2\pi/3$, а остальные шесть двугранных углов равны $\pi/3$.

Отождествим грани этого многогранника по следующей схеме:

$$(A,B,D) \xrightarrow{\varphi_1} (A,E,C);$$
$$(C,A,D) \xrightarrow{\varphi_2} (C,E,B);$$
$$(A,B,E) \xrightarrow{\varphi_3} (C,D,B).$$

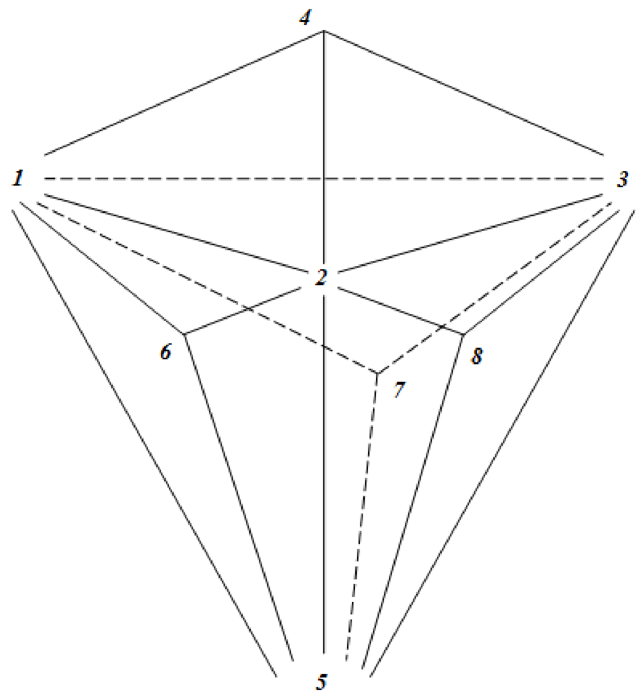


Тогда группа Γ, порожденная движениями $\langle\varphi_1,\varphi_2,\varphi_3\rangle$, будет группой без кручений, а бипирамида $ABCDE$ будет ее фундаментальным многогранником.

Рассмотрим теперь внутри правильного тетраэдра $ABCD$ точку F и разобьем его на четыре симплекса с тремя бесконечно удаленными вершинами и одной собственной вершиной $FABC$, $FCAD$, $FCDB$, $FABC$.

Теперь движением φ_1 "приклеим" тетраэдр $FABD$ к грани (A,E,C) тетраэдра $EABC$ движением φ_2 "приклеим" тетраэдр $FCAD$ к грани
(C,E,B) тетраэдра $EABC$, и движением φ_3^{-1} тетраэдр $FCDB$ "приклеим" к грани (A,B,E) тетраэдра $EABC$, а тетраэдр $FABC$ оставим на месте.

В результате такой переклейки мы получим многогранник с четырьмя вершинами на абсолюте и четырьмя собственными вершинами. Метрика этого многогранника зависит от трех непрерывных параметров (координат точки F).



Кроме того, легко доказать, что за счет выбора точки F этот многогранник можно сделать выпуклым, а все двугранные углы будут иррациональны отностельно π. Грани этого многогранника можно отождествить движениями $\varphi_1, \varphi_2, \varphi_3, \varphi_2\varphi_1^{-1}, \varphi_2\varphi_3, \varphi_1\varphi_3$. Обозначим вершины полученного многогранника цифрами 1, 2, 3, 4, 5, 6, 7, 8. Тогда его грани можно отождествить по схеме:

$$\begin{array}{ll} (1,2,4) \xrightarrow{\varphi_1} (1,5,7) & (1,3,4) \xrightarrow{\varphi_2} (5,3,8) \\ (5,1,6) \xrightarrow{\varphi_3} (2,3,4) & (1,3,7) \xrightarrow{\varphi_2\varphi_1^{-1}} (5,2,8) \\ (1,2,6) \xrightarrow{\varphi_2\varphi_3} (3,2,8) & (5,2,6) \xrightarrow{\varphi_1\varphi_3} (5,3,7) \end{array}$$

Легко доказать, что полученный многогранник будет фундаментальным многогранником группы Γ, а потому разбивает Λ^3 нормально и правильно. С точки зрения метрики множество таких многогранников непрерывно и зависит от трех непрерывных параметров.

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Möbius geometry of the boundary at infinity of complex hyperbolic spaces

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1. Möbius structures and Ptolemy spaces

Two metrics d, d' on a set X are *Möbius equivalent* if for any quadruple $Q = (x, y, z, u) \subset X$ of pairwise distinct points the respective *cross-ratio triples* coincide, $\mathrm{crt}_d(Q) = \mathrm{crt}_{d'}(Q)$, where

$$\mathrm{crt}_d(Q) = (d(x,y) \cdot d(z,u) : d(x,z) \cdot d(y,u) : d(x,u) \cdot d(y,z)) \in \mathbb{R}P^2.$$

We consider *extended* metrics on X for which existence of an *infinitely remote* point $\omega \in X$ is allowed, that is, $d(x, \omega) = \infty$ for all $x \in X$, $x \neq \omega$. We always assume that such a point is unique if exists, and that $d(\omega, \omega) = 0$. We use notation $X_\omega := X \setminus \omega$ and the standard conventions for the calculation with $\omega = \infty$. If ∞ occurs in Q, say $u = \infty$, then $\mathrm{crt}(x, y, z, \infty) = (d(x, y) : d(x, z) : d(y, z))$.

A *Möbius structure* on a set X is a maximal collection $\mathcal{M} = \mathcal{M}(X)$ of metrics on X which are pairwise Möbius equivalent. A topology on X is well defined by a Möbius structure. When a Möbius structure $\mathcal{M}$ on X is fixed, we say that $(X, \mathcal{M})$ or simply X is a *Möbius space.*

A map $f : X \to X'$ between two Möbius spaces is called *Möbius*, if f is injective and for all quadruples $Q \subset X$ of pairwise distinct points

$$\mathrm{crt}(f(Q)) = \mathrm{crt}(Q),$$

where the cross-ratio triples are taken with respect to some (and hence any) metric of the Möbius structures of X, X'. Möbius maps are continuous. If a Möbius map $f : X \to X'$ is bijective, then f^{-1} is Möbius, f is homeomorphism, and the Möbius spaces X, X' are said to be *Möbius equivalent.*

In general different metrics in a Möbius structure $\mathcal{M}$ can look very different. However if two metrics have the same infinitely remote point, then they are homothetic.

A classical example of a Möbius space is the extended $\widehat{\mathbb{R}}^n = \mathbb{R}^n \cup \infty = S^n$, $n \geq 1$, where the Möbius structure is generated by some extended Euclidean metric on $\widehat{\mathbb{R}}^n$. Euclidean metrics which are not homothetic to each other generate different Möbius structures which however are Möbius equivalent.

A Möbius space X is called a *Ptolemy space*, if it satisfies the Ptolemy property, that is, for all quadruples $Q \subset X$ of pairwise distinct points the entries of the respective cross-ratio triple $\mathrm{crt}(Q) \in \mathbb{R}P^2$ satisfies the triangle inequality. The importance of the Ptolemy property comes from the following fact: A Möbius structure $\mathcal{M}$ on a set X is Ptolemy if and only if for all $z \in X$ there exists a metric $d_z \in \mathcal{M}$ with infinitely remote point z.

The classical example of Ptolemy space is $\widehat{\mathbb{R}}^n$ with a standard Möbius structure.

We list some known results on Ptolemy spaces. A real normed vector space, which is ptolemaic, is an inner product space (Schoenberg, 1952); a Riemannian locally ptolemaic space is nonpositively curved (Kay, 1963); all Bourdon and Hamenstädt metrics on $\partial_\infty X$, where X is CAT(-1), generate a Ptolemy space (Foertsch-Schroeder, 2006); a geodesic metric space is CAT(0) if and only if it is ptolemaic and Busemann convex, a ptolemaic proper geodesic metric

[1] This is a joint work with Viktor Shroeder

space is uniquely geodesic (Foertsch-Lytchak-Schroeder, 2007); any Hadamard space ptolemaic, a complete Riemannian manifold is ptolemaic if and only if it is a Hadamard manifold, any Finsler ptolemaic manifold is Riemannian (Buckley-Falk-Wraith, 2009);

A (Ptolemy) circle in a Ptolemy space X is a subset $\sigma \subset X$ homeomorphic to S^1 such that for every quaruple $(x, y, z, u) \in \sigma$ of distinct points the equality

$$|xz||yu| = |xy||zu| + |xu||yz| \tag{1}$$

holds, where it is supposed that the pair (x, z) separates the pair (y, u), i.e. y and u are in different components of $\sigma \setminus \{x, z\}$. Recall the classical Ptolemy theorem that four points x, y, z, u of the Euclidean plane lie on a circle (in this order) if and only if their distances satisfy the Ptolemy equality (1). Let σ be a circle passing through the infinitely remote point ω and let $\sigma_\omega = \sigma \setminus \omega$. Then for x, y, $z \in \sigma_\omega$ (in this order) we have $|xy| + |yz| = |xz|$, i.e. it implies that σ_ω is a geodesic, actually a complete geodesic isometric to $\mathbb{R}$.

A Möbius characterization of the boundary at infinity of real hyperbolic spaces $\partial_\infty \mathrm{H}^{n+1}$ is obtained by T. Foertsch and V. Schroeder, 2009.

Theorem 1. *Let X be a compact Ptolemy space such that through any three points there is a circle. Then X is Möbius equivalent to $\widehat{\mathbb{R}}^n = \partial_\infty \mathrm{H}^{n+1}$.*

2. Ptolemy spaces with many circles and many automorphisms

We are interested in Möbius characterization of the boundary at infinity of rank one symmetric space different from real hyperbolic spaces, for which the answer is given by Theorem 1. Such a boundary is a compact Ptolemy space with many circles and automorphisms, the property, which we formalize in the following four basic axioms. It is convenient to use term a $\mathbb{R}$-*circle* for a Ptolemy circle.

1. Existence axiom: through every two points in X there is a $\mathbb{R}$-circle.
2. Uniqueness axiom: given a quadruple of points $Q \subset X$ such that the Ptolemy equality holds for Q, and three points of Q lie on a $\mathbb{R}$-circle $\sigma \subset X$, then the fourth point of Q lies also on σ.
3. Self-duality axiom: given a $\mathbb{R}$-circle $\sigma \subset X$, let $\psi : (X \setminus \sigma) \times \sigma \to \sigma$ be a map defined by $\psi(x, \omega) \in \sigma$ is the closest to x point in the space X_ω (by Axiom 2, ψ is well defined). Then $\psi(x, \psi(x, \omega)) = \omega$ for all $x \in X \setminus \sigma$, $\omega \in \sigma$.
4. Extension axiom: any Möbius map between any $\mathbb{R}$-circles in X extends to a Möbius automorphism of X.

Conjecture 2. *Let X be a compact Ptolemy space which satisfies Axioms (1)–(4). Then X is Möbius equivalent to the boundary at infinity of rank one symmetric space of noncompact type.*

As an important step towards Conjecture 2, we have the following conjecture. For $\omega \in X$, we consider $X_\omega = X \setminus \omega$ as a metric space with a metric d from the Möbius structure of X with infinitely remote point ω.

Conjecture 3. *Let X be a compact Ptolemy space which satisfies Axiom 1–4. Then for every $\omega \in X$ there is a submetry $\pi_\omega : X_\omega \to B_\omega$ with the base B_ω isometric to an Euclidean space $\mathbb{R}^k$, $k \leq \dim X$, such that any Möbius automorphism $\varphi : X \to X$ with $\varphi(\omega) = \omega'$ induces a homothety $\overline{\varphi} : B_\omega \to B_{\omega'}$ with $\pi_{\omega'} \circ \varphi = \overline{\varphi} \circ \pi_\omega$. Completed fibers $\widehat{F} = F \cup \omega$ of π_ω, called* $\mathbb{K}$-circles*, are homeomorphic to the sphere S^p, $k + p = \dim X$, and the following properties hold*
($1_\mathbb{K}$) through any two distinct points in X there is a unique $\mathbb{K}$-circle;
($2_\mathbb{K}$) any $\mathbb{K}$-circle and any $\mathbb{R}$-circle in X have at most two points in common;

($3_{\mathbb{K}}$) given a $\mathbb{K}$-circle $\widehat{F} = F \cup \omega$ through $\omega \in X$, and $x \in X \setminus \widehat{F}$, there is a unique $\mathbb{R}$-circle $\sigma \subset X$ through x, ω that intersects F;
($4_{\mathbb{K}}$) given distinct $\mathbb{K}$-circles $\widehat{F} = F \cup \omega$, $\widehat{F}' = F' \cup \omega$ through $\omega \in X$ and two $\mathbb{R}$-circles through ω that intersect F, F', for any other $\mathbb{K}$-circle $\widehat{F}'' = F'' \cup \omega$ if F'' intersects one of the $\mathbb{R}$-circles, then it necessarily intersects the other.

This conjecture is much plausible, at the moment we are able to prove all properties ($1_{\mathbb{K}}$)–($4_{\mathbb{K}}$) except the existence in ($3_{\mathbb{K}}$). Our main result is the following.

Theorem 4. *Let X be a compact Ptolemy space which satisfies Axioms (1)–(4). Assume in addition that $p = 1$ in the conclusion of Conjecture 3, that is, X also has properties ($1_{\mathbb{K}}$)–($4_{\mathbb{K}}$) with $p = 1$ and $\mathbb{K} = \mathbb{C}$. Then X is Möbius equivalent to the boundary at infinity of a complex hyperbolic space.*

New tilings of hyperbolic spaces and new manifolds that arise by a manifold reconstruction

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New examples of tilings of hyperbolic spaces H^n $(n = 3, 4, 5)$ and new manifolds that arise by the reconstruction of hyperbolic n-manifolds are given. Geometrical and topological methods are elaborated for the considered cases, with lead, in particular, to the construction of half volume manifolds.

In contrast with Euclidean spaces, the reconstruction of a tile-transitive face-to-face tiling into one non-face-to-face tile-transitive that preserves the shape of the tile for hyperbolic spaces seems to be more difficult. For example, tiling $\mathbb{H}^3$ by regular non-compact octahedra of finite volume admits only several discreet cases of such reconstructions. Below a method of tiling reconstruction is illustrated in more details for 3-dimensional case.

Let consider only parabolic incidences of hyperfaces of the regular octahedron in Thurston's example (hyperbolic structure on the Whitehead links complement in $\mathbb{S}^3$). From fundamental polytope we obtain a manifold with two identical components as a total geodesical boundary, witch represents a sphere with 3 cusps. The symmetry group of these components has the order 12 and, consequently, there are no more possibilities of gluing this component to obtain a manifold, but it should be verified in each case. The geometry of cusps can be seen on the orthogonal orisphere. Finally, the gluing which does not preserve the map given by hyperfaces on boundary, yields non face-to-face incidences on the fundamental polytope. The fundamental groups of the new manifold generate a non-face-to-face (one-time tile-transitive) tiling of the considered hyperbolic space.

In general, if the hyperbolic manifold is constructed by synthetic methods from fundamental polytopes with incidences of hyperfaces, for example from Coxeter polytopes, then we have the possibility to study the total geodesic submanifolds along some hyperfaces, when they does exist. The map of hyperfaces and the symmetry group of these submanifolds are essential in this construction.

In 4-dimensional case we use hyperbolic manifolds obtained from the non-compact regular 24-cells [2, 3, 4]. Like in the case $n = 3$ we select hyperfaces with total geodisical submanifolds situated along them. The reconstruction is done using these submanifolds. Two hyperbolic 4-manifolds, with the cusps over the most symmetric 3-dimensional Euclidean torus, are determinedin in [4]. The new fundamental groups of reconstracted manifolds yield a non-face-to-face tiling of the 4-dimensional hyperbolic space.

Using the geometry of first hyperbolic 5-manifolds [5] of finite volume obtained from given below fundamental polyhedron, non-face-to-faces tilings of $\mathbb{H}^5$ are obtained. The construction of the fundamental polyhedron P^5 in $\mathbb{H}^5$ is important itself.

Let U_{24} be a parabolic bundle in hyperbolic 5-dimensional space $\mathbb{H}^5$ which determines a regular 24-cell on the orthogonal to bundle orisphere Σ^4. The second identic parabolic bundle U'_{24} is taken along the axe of symmetry of U_{24} in the opposite direction to the first bundle. Let also U'_{24} be taken in the dual position to the U_{24}. This is possible because the regular 24-cells is a selfdual polytope. Each bundle, as a rigid solid, can be translated along the axe of symmetry, and we have one metric parameter that controls the dihedral angle of hyperfaces from different bundles. This non-compact polyhedron P^5 of finite volume, telling the hyperbolic space $\mathbb{H}^5$, permits a very simple identification of its 48 hyperfaces, which yields a 5-manifold.

This manifold has the cusps over the most symmetric 4-dimensional Euclidean torus. The stabilizer of any point of the torus is isomorphic to F_4 - the symmetry group of the regular 24-cells. Two-sided embedded total geodesic 4-submanifolds $\mathcal{M}_i^4$ (i=1,48) are found. It is just the factorized symmetry hyperplanes of the polyhedron P^5. Different metric reconstructions of $\mathcal{M}^5$ along $\mathcal{M}^4$ permit to obtain new 5-manifolds. The fundamental polyhedron is not Coxeter and the hyperfaces does not form a total geodesic submanifold, but some of the new manifolds and their fundamental polyhedron permit a reconstruction in order to obtain non-face-to-face tiling of $\mathbb{H}^5$.

All these examples are refereed to non compact polyhedrons and manifolds. For the compact cases we consider two consecutive reconstructions of the 4-dimensional Davis manifold which yields an involution without fixed points on new manifold. The factorization of these manifolds by the above involution gives the complete manifold whose volume is two times less that the volume of the initial manifold. In the communication we will give some other examples of hyperbolic n-manifolds (n=3, 4, 5) $\mathcal{M}^n$ which possess such involution. Some of them are obtained as metrical reconstruction of manifolds described in [7-8].

This topological and geometrical procedure is close related to well known topic of the combinatorial theory of groups: HNN-extension and direct product with common subgroup.

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Examples of neighborly polytopes of dimension D with $D+4$ vertices

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We construct a series of neighborly polytopes using theory of Gale Diagrams.

Let us choose an integer $d \geq 2$. We are going to study polytopes with $n = 2d+4$ vertices which are embedded in $\mathbb{R}^{2d}$.

Definition 1. [1] A polytope in $\mathbb{R}^D$ is called *neighborly* if any subset of its vertices of cardinality $\leq \lfloor D/2 \rfloor$ is a face.

It is shown in [2] that given a polytope in $\mathbb{R}^{2d}$ with $2d+4$ vertices, one can construct an (affine) Gale diagram, i. e. a set of black and white points (in this particular case) in $\mathbb{R}^2$ that are in one-to-one correspondence with the vertices of the polytope. This diagram completely describes the combinatorics of the polytope. For example, to determine whether the vertices $v_1, \dots, v_k$ of the polytope form a face consider the points $p_{k+1} \dots, p_n$ of the diagram corresponding to *the rest of the vertices*. Let P be the convex hull of all black points among $p_{k+1}, \dots, p_n$ and let Q be the convex hull of all of the white ones. Then the vertices $v_1, \dots, v_k$ form a face iff the intersection of the relative interiors of P and Q is not empty. Affine Gale diagram of a polytope is defined up to projective equivalence.

We impose one more restriction on polytopes we are going to study: each of them should have an affine Gale diagram with exactly $d+3$ white points, and these points should form a convex polygon.

Definition 2. A set S of $2d+4$ black and white points in $\mathbb{R}^2$ is called a *T-diagram* if

1. Exactly $d+3$ of the points in S are black and they form a convex polygon.

2. S is an affine Gale diagram of a neighborly polytope.

Definition 3. A neighborly polytope is called *T-polytope* if it has an affine Gale diagram which is T-diagram.

Proposition 1. *The following conditions are equivalent:*

1. S is a T-diagram.

2. S is a set of $d+3$ black points and $d+1$ white points, black points form a convex polygon and there is exactly one white point inside each triangle with black vertices.

Theorem 1. *Two T-diagrams are combinatorially equivalent iff the corresponding T-polytopes are combinatorially equivalent.*

T-diagrams can be enumerated using 3-trees that are defined as follows:

Definition 4. A tree with the additional structure is called a *3-tree* if

1. Each of its vertices is of degree 1 or 3.

2. (Additional structure) For each vertex A of degree 3, the edges incident to A are cyclically ordered. These cyclic orderings induce cyclic order on the set of vertices of degree 1.

Informally speaking, a 3-tree is a tree with vertices of degree 3 or 1 that is embedded in $\mathbb{R}^2$ where an orientation is chosen.

Definition 5. A 3-tree is called *the characteristic tree* of a T-diagram if there exists a one-to-one correspondence φ between the points of the diagram and the vertices of the 3-tree such that

1. The white points correspond to the vertices of degree 3, the black points correspond to the vertices of degree 1.
2. The cyclic order on the vertices of degree 1 corresponds to the natural cyclic order of the vertices of the (convex) $(d+3)$-gon for some choice of orientation of the $\mathbb{R}^2$ containing the diagram.
3. A white point B is inside a triangle with three black vertices A_i, A_j, A_k iff any two of the three paths connecting $\varphi(B)$ with $\varphi(A_i), \varphi(A_j), \varphi(A_k)$ have no common edges.

Theorem 2. *1. Given a T-diagram, there exists a characteristic tree. It is unique up to the simultaneous inverting of the cyclic orders at the vertices of degree 3.*

2. Given a 3-tree R with at least 8 vertices, there exists a unique (up to combinatorial equivalence) T-diagram with characteristic tree R.

This enumeration theorem yields the following formula:

Theorem 3. *Given d, the number of T-polytopes in $\mathbb{R}^{2d}$ is*

$$\frac{T_{d+1}}{2(d+3)} + \frac{3T_{(d+3)/2-1}}{4} + \frac{T_{(d+3)/3-1}}{3} + \frac{T_{d/2}}{2},$$

where

$$T_x = \begin{cases} 0, & x \notin \mathbb{N} \\ C_{2x}^{x}/(x+1), & x \in \mathbb{N}. \end{cases}$$

We see that this number grows exponentially as d grows.

Theorem 4. *Given d and m, the number of faces of dimension m that contain a vertex A of a T-polytope in $\mathbb{R}^{2d}$ depends on d and m but does not depend on the polytope and the vertex A.*

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On the Covering Theorem of Bezicovich

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The aim of my talk is to tell about some new generalizations of the classical covering theorem of Besicovich.

Theorem of Besicovich. If $\mathcal{F} = \{C(x, r_x)\}_{x\in A}$ is a family of cubes in $\mathbb{R}^n$ such that $\sup_{x\in A} r_x \leq \infty$, and A is a set in $\mathbb{R}^n$, then there exists $A_0 \subseteq A$ such that following conditions hold:
(1) $A \subseteq \bigcup_{x\in A_0} C(x, r_x)$; (2) $\mathrm{mult}(\{C(x, r_x)\}_{x\in A_0}) \leq a(n)$;
(3)

$$\{B(x, r_x)\}_{x\in A_0} = \bigcup_{i=1}^{b(n)} \mathcal{F}_i,$$

where $\mathcal{F}_i$ is a packing for any i; (4) $a(n)$ and $b(n)$ depends on n only.

This theorem plays an important part in the theory of functions and the measure theory (see [5]).

Let M be a finite-dimensional space with a metric ρ_M of a lower bounded curvature $\mu > -\infty$ in sense of Aleksandrov (FDSLC).

Definitions. A family of balls $\{B(x, r_x)\}_{x\in E}$ in M is called a covering of Bezicovich of a set $E \subseteq M$. A family of disjoint sets is called a packing. Suppose $\mathcal{F}$ is a family of subsets of a set E, then by definition, put

$$\mathrm{mult}(\mathcal{F}) = \sup_{x\in E} \mathrm{mult}(x, \mathcal{F}) = \sup_{x} |\{V \in \mathcal{F} : x \in V\}|.$$

The value $\mathrm{mult}(x, \mathcal{F})$ is called a multiplicity of family $\mathcal{F}$ in the point x, and $\mathrm{mult}(\mathcal{F})$ is called a multiplicity of family $\mathcal{F}$.

By L^n_μ denote n-dimensional hyperbolic space, by E^n denote Euclidean space.

Covering Theorem. If $\{B(x, r_x)\}_{x\in E}$ is a covering of Bezicovich of a subset E in a complete n-dimensional space of a lower bounded curvature $\mu > -\infty$ in sense of Aleksandrov M such that $sup_{x\in E} r_x < \infty$, then there exists $E_0 \subseteq E$ such that following conditions hold:

1. $E \subseteq \bigcup_{x\in E_0} B(x, r_x)$;
2. $\mathrm{mult}(\{B(x, r_x)\}_{x\in E_0}) \leq a(n, r, \mu)$, where $a(n, r, \mu)$ depends on n, r, μ only;
3. $$\{B(x, r_x)\}_{x\in E_0} = \bigcup_{i=1}^{b(n,r,\mu)} \mathcal{F}_i,$$

   where $\mathcal{F}_i$ is a packing for any i, and $b(n, r, \mu)$ depends on n, r, μ only.

Remark 1. If we take a covering of Bezicovich $\{B(n, n)\}_{n\in\mathbb{N}}$ in $\mathbb{R}$, then we see that the condition $\sup_{x\in E} r_x < \infty$ is essentially.

Remark 2. If $\mu \geq 0$, then $a(n, r, M)$ and $b(n, r, M)$ it possible to take independent of r. If $\mu < 0$, then we can not have do it.

Definition. A family $\mathcal{F}$ of compact convex sets in $\mathbb{R}^n$ is called absolute monotone if for any $V,\ W \in P$ there exists U such that $W = V + U$, (where $V + U$ denote Minkowski sum) or and vice versa. Denote by $V \leq W$, if $W = V + U$. If there exists V_0 such that $V + x \subset V_0$ for any $V \in \mathcal{F}$ and some $x \in \mathbb{R}^n$, then we say that a family $\mathcal{F}$ is bounded.

It is clear that a monotone family $\mathcal{F}$ of parallelotopes and a family of homothets of a convex set is absolute monotone. **Theorem.** Suppose $\mathcal{E}$ is a bounded absolute monotone family of centrally symmetric convex compact sets in $\mathbb{R}^n$ with center in 0, $A \subset \mathbb{R}^n$ and $\mathcal{F} = \{B_x + x\}_{x \in A, B_x \in \mathcal{E}}$. Then there exists $A_0 \subseteq A$ such that following conditions hold:
(1) $A \subseteq \bigcup_{x \in A_0} B_x + x$; (2) $\mathrm{mult}(\{B_x + x\}_{x \in A_0}) \leq a(n)$;
(3)

$$\{B_x + x\}_{x \in A_0} = \bigcup_{i=1}^{b(n)} \mathcal{F}_i,$$

where $\mathcal{F}_i$ is a packing for any i; (4) $a(n)$ and $b(n)$ depend on n only.

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Minkowski sums of Voronoi Polytopes and Commensurate Delone Tilings

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It is natural to ask whether the Voronoi polytope for a lattice can be written as the Minkowski sum of polytopes that are also Voronoi polytopes for lattices. S. S. Ryshkov answered this question by showing that the Voronoi polytope for a point in the relative interior of an L-type is affinely equivalent to a weighted Minkowski sum of Voronoi polytopes for the edge forms of the L-type.

I will initiate proceedings by sketching the proof of the Lemma: A Voronoi polytope can be written as the Minkowski sum of Voronoi polytopes if and only if the Delone tilings for the two summands are commensurate (a notion that will be defined during the lecture). This Lemma serves as a corner stone for a beautiful duality theory that relates commensurate Delone tilings and the Minkowski decomposition of Voronoi polytopes; it also provides the key step in proving Ryshkov's Theorem. The line of argument I use will parallel that used in a preliminary version of the duality theory relating dicings and Voronoi zonotopes.

Delone Sets - Diffraction and the Cut-and-Project Method

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A point set D in $\mathbb{R}^d$ is *uniformly discrete*, if there is $r > 0$ such that each ball of radius r contains at most one point of D. D is called *relatively dense*, if each ball of radius R contains at least one point of D. Delone sets are point sets in $\mathbb{R}^d$ which are uniformly discrete and relatively dense. They were introduced by B.N. Delone in order to study algebraic problems by using geometric tools [2].

If you do a google search (or a MathSciNet search) for "Delone set"today, almost all hits have to do with mathematical quasicrystals. Mathematical quasicrystals are Delone sets in $\mathbb{R}^d$ which are not periodic (i.e., they don't possess any translational symmetry), but show a high degree of local and global order. There is no generally agreed rigorous definition of "quasicrystal"today. However, any expert would agree that for instance the vertex set of the famous Penrose tiling form a mathematical quasicrystal.

Real (physical) quasicrystals were found in the 1980s by their diffraction spectrum: They show a pure point diffraction spectrum (like a crystal), but they show also 8-fold, 10-fold or 12-fold symmetry, which is impossible for crystals (i.e., structures with translational symmetry). This discovery induced a lot of experimental and theoretical research, leading to the development of a mathematical theory of quasicrystals.

Today the mathematics of Delone sets with a pure point diffraction spectrum is pretty well understood. A central result is the following, obtained by Hof [4] for the Euclidean case and generalized by Schlottmann [6].

Theorem 1 (Hof, Schlottmann)**.** *Each regular* model set *(or cut-and-project set) has pure point diffraction spectrum.*

A Delone set in $\mathbb{R}^d$ is a model set, if it can be obtained by a projection from some higher dimensional space in the following way:

$$\begin{array}{ccccc} \mathbb{R}^d & \xleftarrow{\pi_1} & \mathbb{R}^d \times H & \xrightarrow{\pi_2} & H \\ \cup & & \cup & & \cup \\ D & & \Lambda & & W \end{array}$$

- H a locally compact Abelian group
- Λ a lattice in $\mathbb{R}^d \times H$
- π_1, π_2 are projections, such that $\pi_1|_\Lambda$ is injective, and $\pi_2(\Lambda)$ is dense
- The *window* W is compact ($\mu(W) = 0$)

Then $D = \{\pi_1(x) \,|\, x \in \Lambda, \pi_2(x) \in W\}$ is a (regular) model set.

In turn, model sets have been studied already in the 70s by Meyer [5]. It was shown that each model set is a Meyer set, and each Meyer set is a subset of a model set. A Meyer set is a Delone set D such that $D - D$ is uniformly discrete.

It is clear from the definitions that each lattice in $\mathbb{R}^d$ is a model set as well as a Meyer set. Thus both model sets and Meyer sets can be regarded as generalisations of lattices.

Within this context, one can ask a lot of interesting questions. Here we focus on three of them.

(a) Are there Meyer sets which are not pure point diffractive?

(b) Are there pure point diffractive sets which are not model sets?

(c) Has each model set an average lattice?

The answers are (a) yes, for instance certain selfsimilar subsets of $\mathbb{Z}^d$ [3]; (b) yes, the set of visible lattice points (which is not Delone) [1]; (c) yes in dimensions one and two, unknown in general (work in progress, together with N. Dolbilin and A. Garber).

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Combinatorial diameter of parallelohedra[1]

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One of the most important and well known conjectures in the modern theory of parallelohedra is the Voronoi conjecture.

The Voronoi Conjecture. *Every parallelohedron in $\mathbb{R}^d$ is affinely equivalent to a Dirichlet-Voronoi domain of some d-dimensional lattice.* The Voronoi conjecture was proved for some particular families of parallelohedra by Voronoi, Zhitomirski and Ordin.

In all of the mentioned results proof of the Voronoi conjecture uses the methods of canonical scaling and positive quadratic forms. The *canonical scaling* is a rule which we use to associate a real number to any $(d-1)$-face of a polytope to satisfy some relation on this numbers for any neighborhood of $(d-2)$-dimensional face of tiling of space $\mathbb{R}^d$.

In this talk we will discuss a way how to find a value of canonical scaling for arbitrary parallelohedron in the shortest possible way. We will try to find the *combinatorial diameter* of the parallelohedron, i.e. the diameter of the Venkov graph of the parallelohedron. *Venkov graph* of the parallelohedron P is the graph whose vertices are pairs of opposite facets P and two pairs are connected with an edge if and only if some facets from these pairs have a common $(d-2)$-face.

We will prove the following

Theorem. *If parallelohedron P is d-dimensional zonotope, i.e. P is a d-dimensional Minkowski sum of line segments, then the combinatorial diameter of P is not greater than $\lceil \log_2 d \rceil$.*

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О проблеме Вороного для параллелоэдров

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Понятие параллелоэдра и сам термин были введены кристаллографом Е.С. Фчдоровым [1] (1885) как одно из основных понятий кристаллографии. *Параллелоэдр* d измерений определяется как выпуклый эвклидов многогранник, который своими параллельными копиями разбивает пространство $\mathbb{E}^d$ нормальным образом, то есть если пересечение двух многогранников не пусто, то оно есть их общая целая грань некоторой размерности.

Одна из основных задач теории параллелоэдров — нахождение алгоритма, перечисляющего для данной размерности все комбинаторные типы параллелоэдров — до сих пор остачтся нерешчнной. Г.Х. Вороной [6] построил теорию параллелоэдров Дирихле-Вороного, в которой приведчн алгоритм перечисления всех комбинаторных типов параллелоэдров Вороного. Он высказал гипотезу о том, что любой параллелоэдр аффинно-эквивалентен некоторому параллелоэдру Дирихле-Вороного. Под *параллелоэдром Дирихле-Вороного* мы понимаем область в $\mathbb{E}^d$, построенную по некоторой d-мерной решчтке $\Lambda \subset \mathbb{E}^d$ и состоящую из точек, для которых данная точка $p \in \Lambda$ является ближайшей среди всех точек решчтки. Нетрудно показать, что такая область является d-мерным многогранником в $\mathbb{E}^d$ и действительно является параллелоэдром.

Вороной доказал свою гипотезу для, так называемых, примитивных параллелоэдров. Позднее О.К. Житомирский [7] усилил теорему Вороного, доказав гипотезу для очень широкого класса — для примитивных в $(d-2)$-мерных гранях (это ограничение эквивалентно тому, что в каждой $(d-2)$-мерной грани сходятся ровно 3 параллелоэдра). R. Erdahl [8] доказал гипотезу Вороного для параллелоэдров, которые являются зоноэдрами. Последние результаты были получены А. Ординым [9] для, так называемых, *3-неразложимых (3-irreducible)* параллелоэдров.

В доказательствах теорем Вороного, Житомирского, Ордина происходит построение так называемой канонической нормировки:

Пусть T — разбиение пространства $\mathbb{E}^d$ на параллельные копии данного параллелоэдра P_0. Обозначим через $\mathcal{F}^i$ множество всех i-мерных граней этого разбиения.

Определение. Пусть S^{n-1} — произвольное множество гиперграней: $S^{n-1} \subseteq \mathcal{F}^{n-1}$ разбиения T. *Канонической нормировкой* S^{n-1} называется такая функция $s : S^{n-1} \to \mathbb{R}_+$, что

1. Если $F_1, F_2, F_3 \in S^{n-1}$ — гиперграни, сходящиеся в примитивной $(d-2)$-грани, тогда для некоторого (так называемого согласованного) выбора направлений единичных нормалей $\mathbf{n}_i$ к этим граням $s(F_1)\mathbf{n}_1 + s(F_2)\mathbf{n}_2 + s(F_3)\mathbf{n}_3 = 0$

2. Если $F_1, F_2, F_3, F_4 \in S^{n-1}$ — гиперграни, сходящиеся в непримитивной $(d-2)$-грани, тогда для некоторого (так называемого согласованного) выбора единичных нормалей $\mathbf{n}_i$ к этим граням $s(F_1)\mathbf{n}_1 + s(F_2)\mathbf{n}_2 + s(F_3)\mathbf{n}_3 + s(F_3)\mathbf{n}_4 = 0$

Ключевым моментом доказательств перечисленных выше теорем является построение особой полиэдральной поверхности. Она строится как график кусочно-линейной функции $G : \mathbb{E}^d \to \mathbb{R}$, так называемой *женератрисы Вороного*, которая строго линейна на каждом параллелоэдре разбиения T. С помощью канонической нормировки определяются приращения градиентов на смежных гранях данной полиэдральной поверхности.

В случаях Вороного, Житомирского, Ордина (каждый последующий охватывает строго более широкий класс параллелоэдров) по построенной женератрисе определяется единственным образом вписанный в еч график эллиптический параболоид. Этот параболоид аффинным преобразованием можно перевести в параболоид сферический. С помощью оптических свойств сферического параболоида показывается, что полученное из разбиения T при данном аффинном преобразовании разбиение T' является разбиением на параллелоэдры Дирихле-Вороного, и гипотеза в соотвествующих случаях верна.

Автором доказана теорема, утверждающая, что указанное построение функции женератрисы и соответствующей полиэдральной поверхности возможно для любой канонической нормировки (а не только в случаях Вороного, Житомирского, Ордина). В то же время приведчн пример канонической нормировки (и соответствующей поверхности), не описанной возле какого-либо эллиптического параболоида. Из этих теоремы и примера следует, что существования канонической нормировки для некоторого параллелоэдра может быть недостаточно для доказательства гипотезы Вороного (как было во всех приведчнных выше случаях).

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Approximation of convex sets by projections of convex polyhedra

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The problem of approximation of convex sets by polyhedra with a given precision ε is applied, in particular, for convex programming problems (CP) of the form: $f_0(x) \to \min$, $f_i(x) \leqslant 0$, $i = 1, \dots, m$, $x \in \mathbb{R}^n$ (f_i are convex functions), i.e., finding minima of a convex function f_0 on a convex domain G, given by a system of convex constraints. It is generally known that for large dimensions n such problems may have enormous compexity. The only exceptions are linear programming (LP) problems (f_0 and f_i are all linear), which are efficiently solvable even for $n > 10^6$ [1]. Thus, if one approximates the graph of f_0 and the domain G by polyhedra with the precision ε, then the CP problem is approximately reduced to the corresponding LP problem, which can then be efficiently solved.

The problem of approximation of convex sets by polyhedra in the Hausdorff metric has been thoroughly studied in the literature, see [2] and references therein. It is known that the number of hyperfaces (i.e., linear constraints) of the aproximating polyhedron is bounded above by $C\varepsilon^{\frac{1-n}{2}}$, where n is the dimension, and this estimate is sharp even for the Euclidean ball. We analyze a new approach that reduces esseintially the number of faces for some classes of convex domains. This approach is based on the following simple fact: an orthogonal projection of a convex polyhedron can have much more faces than the original polyhedron. Therefore, this is reasonable to approximate convex sets not be polyhedra in the original space, but by projections of polyhedra in a larger space.

The idea of applying projections from larger dimensions was first proposed by A. Ben-Taal and A. Nemirovsky for approximation of quadrics in the Hausdorff metric [3]. We extend this approach to a wider class of convex sets. The dimensions of the approximating polyhedra are of order $C\left(\ln \frac{1}{\varepsilon}\right)^k$, and the number of faces is usually the double of this number.

For generalizing this method to a wider class of convex functions we introduce another approach. The idea is to approximate first graphs of univariate convex functions, by polygons, and then, using a special inductive procedure, reduce the multivariate problem to the univariate one. The corresponding polyhedron of a larger dimension is constructed in the inductive procedure. The complexity of the algorithm increases to a polynomial one, since the number of sides of the approximating polygons is estimated as $C\frac{1}{\sqrt{\varepsilon}}$. However, this enlarges the class of approximated functions and sets. We present an optimal (by the number of sides N_ε) algorithm for univariate approximation by polygons, and elaborate a special inductive procedure to pass to multivariate case. This allows us to approximate, for instance, the unit balls of the L_p-norm, the graph of the enthropy function, etc.

This approach of approximation by projections makes it possible, in particular, to solve CP problems with the functions of those special classes, reducing them to LP problems. The efficiency of this method is illustrated with several examples.

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The Delaunay tiling and the conjecture of Voronoi

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A *parallelotope* is a polytope translation copies of which fill the space without gaps and intersections by inner points. A special case of a parallelotope is the *Voronoi polytope* of a point lattice. It is the closure of all points of space that are nearer to a given point of the lattice than to other points of it.

The famous conjecture of Voronoi asserts an affine equivalence of a parallelotope to a Voronoi polytope. This conjectutre is equivalent to an existence of a dual tiling for the tiling onto parallelotopes.

According to general theory of tilings onto n-dimensional polytopes, two tilings are *dual* if there is a one-to-one correspondence between k-faces of one tiling and $(n-k)$-faces of the other one, such that the affine spaces of corresponding faces are orthogonal. It is proved in the general theory of tilings that a primitive tiling and a tiling onto zonotopes have dual tilings (see, for example, [1]). In the first case, the dual tiling consists of simplices. In the second one, the dual tiling is formed by an arrangement of hyperplanes that are orthogonal to edges of the zonotopes. This implies that the Voronoi conjecture is true when parallelotopes are primitive or are zonotopes.

Voronoi calls a parallelotope *canonically defined* if it is affinely equivalent to a Voronoi polytope. A tiling onto canonically defined parallelotopes has two types of dual tilings: the above defined dual tiling and a topolgically dual tiling. In topologically dual tilings, spaces of corresponding faces need not to be orthogonal. It is shown in this talk that a tiling onto parallelotopes has a *topologically dual* tiling onto Delaunay polytopes.

It is well known, that if parallelotopes of a tiling are Voronoi polytopes, then the both mutually dual tilings exist and coincide. Polytopes of the dual tiling are called Delaunay polytopes. A Delaunay polytope corresponds to a vertex v of the tiling onto Voronoi polytopes. It is the convex hull of the centers of all Voronoi polytopes having v as a vertex.

It is naturally to define similarly Delaunay polytopes of the tiling onto parallelotope. Ákos Horváth proved in [2] that the centers of all parallelotopes having a common vertex are vertices of the corresponding Delaunay polytope. But he did not prove that the such defined Delaunay polytope is full dimensional. In this talk, this gap is filled.

An n-dimensional parallelotope $P=P(0)$ with its center in origin can be described as follows.

$$P=\{x\in\mathbf{R}^n:-\frac{1}{2}p_i^Tt_i\le p_i^Tx\le\frac{1}{2}p_i^Tt_i,\ i\in I_P\}.$$

Here I_P is the set of indices of pairs of opposite facets. The vector t_i connects the center of $P(0)$ with the center of the parallelotope $P(t_i)$ that is adjacent to $P(0)$ by a facet F_i. The vector p_i is the facet vector of F_i.

The edges of the above defined Delaunay polytopes are parallel and equal by norm to vectors t_i. If a parallelotope is defined *canonically* then there is a linear map $x\to Qx$ such that $p_i=Qt_i$ for all $i\in I_P$. In this case, the facet vectors p_i are called canonically defined.

The map Q transforms each Delaunay polytope P_D into canonically defined Delaunay polytope QP_D. The edges of canonically defined Delaunay polytopes are parallel and equal by norm to canonically defined facet vectors p_i. The canonically defined Delaunay polytopes form the dual tiling for the tiling onto canonically defined parallelotopes.

I show how simple can be transformed a simplicial Delaunay polytope into canonically defined simplex.

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The cut loci and the conjugate loci on n dimensional ellipsoids[1]

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The notion of cut locus, introduced by H. Poincaré in 1905, gain an important place in global riemannian geometry. The *cut locus* $C(x)$ *of* the point x in the riemannian manifold M is the set of all extremities (different from x) of maximal (with respect to inclusion) shortest paths starting at x.

The study of cut locus has long history, there are many remarkable results. S. B. Myers for $n = 2$, and M. Buchner for general n, established that the cut locus of a real analytic riemannian manifold of dimension n is homeomorphic to a finite $(n-1)$-dimensional simplicial complex.

But in most cases, to determine cut loci are quite difficult problems. There are only a few cases where the cut loci are well understood; for example, symmetric spaces and some homogeneous spaces (T.Sakai and M.Takeuchi), certain surfaces of revolution (M.Tanaka ans R.Sinclair), the tri-axial ellipsoids and some Liouville surfaces (K.Kiyohara and I.) Especially in higher dimensional case there are not any results without symmtric spaces and some singular spaces (Vilcu and I.), even if quadric hypersurfaces.

In this talk, we determine the cut locus and the conjugate locus of the ellipsoid M : $\sum_{i=0}^{n} u_i^2/a_i = 1$ $(0 < a_n < \cdots < a_0)$, Let J_k be the submanifolds of M defined by

$$J_k = \{u \in M \mid u_k = 0, \quad \sum_{i \neq k} \frac{u_i^2}{a_i - a_k} = 1\ \} \qquad (1 \leqslant k \leqslant n-1)$$

Let $(\lambda_1^0, \ldots, \lambda_n^0)$ be the elliptic coordinates of p.

Theorem 1. *If $p \notin J_{n-1}$, then $C(p)$ is an $(n-1)$-dimensional closed disk which is contained in a submanifold (possibly with boundary) defined by $\lambda_n = \lambda_n^0$.*

If $p \in J_{n-1}$, then $C(p)$ is an $(n-2)$-dimensional closed disk contained in J_{n-1}. It is identical with the cut locus of p in the $(n-1)$-dimensional ellipsoid N_{n-1}.

Let K_i be the i-th conjugate locus. For the singularities of K_i, we have the following theorem.

Theorem 2. *Let p be a point with $u_i \neq 0$ $(\forall i)$. Then the set of singularities of the first conjugate locus K_1 of p consisits of three connected components; one of them are diffeomorphic to $S^{n-2}\times$ a cusp curve, and the interior of the other two are diffeomorphic to (immersed) $D^{n-2}\times$ a cusp curve.*

Here, D^{n-2} denotes the $(n-2)$-open disk. The boundary of the latter components are still unclear up to now. This theorem is a higher dimensional version of the so-called Jacobi's last geometric statement: The first conjugate locus of a general point on a two-dimensional ellipsoid contains exactly four cusps.

Moreover, we have the following theorem, if the ellipsoid M is enough close to the round sphere.

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[1]This is a joint work with K. Kiyohara, Okayama Univ.

Theorem 3. *The set of singularities of K_i $(2 \leq i \leq n-2)$ consists of two conected components whose interiors are diffeomorphic to*

$$S^{n-1-i} \times D^{i-1} \times \text{ a cusp curve }, \quad D^{n-1-i} \times S^{i-1} \times \text{ a cusp curve}$$

respectively. K_{n-1} is similar to K_1. The intersection $K_i \cap K_{i+1}$ is identical to the common boundary of their singularity sets; $S^{n-2-i} \times S^{i-1}$.

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Random Sequential Packing of Cubes[1].

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Because of analytical difficulties for higher dimension, one-dimensional random sequential packing has received attention (Renyi (1958)), Itoh (1980)). The one-dimensional model can be extended to the random sequential packing of cubes. Consider the random sequential packing of cubes of edge length 1 in a parallel position in a larger cube of edge length x. It seems to be natural to expect for the d-dimensional extension that the limiting packing density (i) exists and (ii) is equal to β^d as x tends to ∞, where β is the limiting packing density for $d = 1$ given by Renyi (1958), which is called Palasti' s conjecture. The conjecture (i) is shown by Penrose (2001). It is known that the computer simulations do not support the conjecture (ii).

Consider the simplest random sequential packing with rigid boundary, i.e. a packing in which cubes of edge length 2 are put sequentially at random into the cube of edge length 4, with a cubic grid with unit edge length, in a parallel position on the grid . Consider the packing density γ_d of dimension d. The computer simulations up to dimension 11 (Itoh and Ueda (1983), Itoh and solomon (1986)) seems to fit to $\gamma_d = d^{-\alpha}$ with an appropriate constant α.

The expected number of decrease of the packing density is shown to be less than $(\frac{4}{3})^d$ at each step of the random sequential packing (Poyarkov (2004, 2007) , which proves that the expected number of cubes at the saturation is larger than $(\frac{3}{2})^d$.

Consider the simple random sequential packing with periodic boundary (random sequential packing into torus). The case $d = 1, 2$ gives the tiling of cubes (100 per cent packing density), while the case $3 \leqslant d$ does not always give the tiling of cubes. We study geometrical structure generated by of packing of cubes (Dutour Sikiric, Itoh and Poyarkov (2007)).

Such a cube packing is called *non-extendible* if we cannot insert a cube in the complement of the packing. In dimension 3, there is a unique non-extendible cube packing with 4 cubes. We prove that d-dimensional cube packings with more than $2^d - 3$ cubes can be extended to cube tilings. We also give a lower bound on the number N of cubes of non-extendible cube packings.

Given such a cube packing and $z \in \mathbf{Z}^d$, we denote by N_z the number of cubes inside the 4-cube $z + [0, 4[^d$ and call *second moment* the average of N_z^2. We prove that the regular tiling by cubes has maximal second moment and give a lower bound on the second moment of a cube packing in terms of its density and dimension.

We consider sequential random packing of cubes $z + [0, 1]^n$ with $z \in \frac{1}{N}\mathbf{Z}^n$ into the cube $[0, 2]^n$ and the torus $\mathbf{R}^n/2\mathbf{Z}^n$ as $N \to \infty$. In the cube case $[0, 2]^n$ as $N \to \infty$ the random cube packings thus obtained are reduced to a single cube with probability $1 - O\left(\frac{1}{N}\right)$. In the torus case the situation is different: for $n \leqslant 2$, sequential random cube packing yields cube tilings, but for $n \geqslant 3$ with strictly positive probability, one obtains non-extensible cube packings.

So, we introduce the notion of combinatorial cube packing, which instead of depending on N depend on some parameters (Dutour Sikiric and Itoh (2010)). We use use them to derive an expansion of the packing density in powers of $\frac{1}{N}$. The explicit computation is done in the cube case. In the torus case, the situation is more complicate and we restrict ourselves to the case $N \to \infty$ of strictly positive probability.

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Infinitesimal rigidity of convex polyhedra and the discrete Hilbert-Einstein functional

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We discuss a connection between the following two theorems.

Theorem 1. *Let $P \subset \mathbb{R}^3$ be a convex polyhedron. Then P is infinitesimally rigid.*

In other words, assume that every vertex of P moves with a constant velocity so that all edge lengths don't change in the first order (assume for simplicity all faces to be triangles). Then the whole polyhedron is subject to an infinitesimal rigid motion.

Theorem 2. *Let $Q \subset \mathbb{R}^3$ be a convex polyhedron. Then Q is Minkowski rigid.*

In other words, assume that the planes of all of its faces are infinitesimally translated so that the area of each face remains constant in the first order. Then Q is infinitesimally translated as a rigid body.

I know of three ways to relate Theorems 1 and 2, and will concentrate on the third one.

1. Theorems 1 and 2 can be proved by the same argument, counting the number of sign changes around a vertex, or along the boundary of a face, respectively.
2. Infinitesimal rigidity of a polyhedron P can be derived from its Minkowski rigidity, by associating to an infinitesimal rotation of a face a parallel translation of it by the normal component of the rotation vector.
3. Infinitesimal rigidity of P is equivalent to Minkowski rigidity of its polar dual, as described below.

Theorem 2 can be proved as follows. Choose a coordinate origin inside Q. For every face of Q, consider its support number h_i (distance from the origin to the plane of the face) and its area F_i. Infinitesimal translations of faces can be described by variations of the support numbers. Therefore Theorem 2 is equivalent to

$$\dim \ker \left(\frac{\partial F_i}{\partial h_j} \right) = 3, \tag{2}$$

where the kernel is generated by translations of Q as a rigid body. The equation (2) is proved in the mixed volumes theory.

This approach can be dualized in the context of Theorem 1. Choose a coordinate origin inside P. For every vertex of P, consider its distance r_i from the origin. An infinitesimal isometric deformation of P results in variations of (r_i). Let us see which variations of (r_i) can appear in this way. For this, decompose P into pyramids with apex at the origin and faces of P as bases. Every variation of (r_i) induces infinitesimal deformations of pyramids. This results in an infinitesimal isometric deformation of P if and only if the sum ω_i of dihedral angles around the edge joining i-th vertex to the origin remains constant in the first order. That is, infinitesimal isometric deformations correspond to elements in the kernel of $\left(\frac{\partial \omega_i}{\partial r_j}\right)$. Therefore Theorem 1 is equivalent to

$$\dim \ker \left(\frac{\partial \omega_i}{\partial r_j} \right) = 3, \tag{3}$$

where the kernel consists of variations that come from moving the origin.

Now, an equivalence between Theorems 1 and 2 is established through a striking identity

$$\frac{\partial \omega_i}{\partial r_j}(P) = -\frac{\partial F_i}{\partial h_j}(P^*), \tag{4}$$

where P^* is the polar dual to P with respect to the origin.

A further interpretation of these arguments is possible. Note that

$$F_i = \frac{\partial \operatorname{vol}(Q)}{\partial h_i}.$$

Therefore the matrix $\left(\frac{\partial F_i}{\partial h_j}\right)$ is symmetric and corresponds to the second variation of the volume of Q. In the dual setting, we have the discrete Hilbert-Einstein functional

$$S(P) = \sum_i r_i(2\pi - \omega_i) + \sum_{ij} \ell_{ij}(\pi - \theta_{ij}),$$

where ℓ_{ij} and θ_{ij} are the length of, respectively the dihedral angle at, the edge ij of P. The equation

$$2\pi - \omega_i = \frac{\partial S(P)}{\partial r_i}$$

follows from the Schläfli formula.

The duality between the Hilbert-Einstein functional and the volume of the dual is more apparent in the hyperbolic geometry. The dual object to a convex hyperbolic polyhedron is a convex polyhedron in the de Sitter space.

The chromatic number of a normed space

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This work concerns the classical Nelson – Hadwiger problem which consists in finding the value $\chi(\mathbb{R}^n)$ equal to the minimum number of colors needed to paint all the points in $\mathbb{R}^n$ so that any two points at distance 1 apart receive different colors (see [2]). The quantity $\chi(\mathbb{R}^n)$ is called *the chromatic number of* $\mathbb{R}^n$.

The history of the Nelson – Hadwiger problem can be found in many books and surveys (see, e.g., [3], [6]).

An important variant of the problem was proposed for normed spaces $\mathbb{R}^n_K$ with norms induced by arbitrary centrally symmetric convex bodies K. Let $\chi(\mathbb{R}^n_K)$ be the corresponding chromatic number. In [4], the authors obtained the estimate

$$\chi(\mathbb{R}^n_K) \leq cn(\ln n)5^n$$

with some constant c, which does not depend on K. Here we substantially improve this result.

Theorem 1. *The inequality holds*

$$\chi(\mathbb{R}^n_K) \leq \frac{(\ln n + \ln\ln n + \ln 4 + 1 + o(1))}{\ln\sqrt{2}} \cdot 4^n.$$

Further improvements of the bound in Theorem 1 are possible in the case of the l_p-space $\mathbb{R}^n_p$.

Theorem 2. *The inequality holds*

$$\chi(\mathbb{R}^n_p) \leq 2^{(1+c_p+\delta_n)n},$$

where $\delta_n \to 0$ *with* $n \to \infty$, $c_p < 1$ *for* $p > 2$, *and* $c_p \to 0$ *with* $p \to \infty$.

To prove these theorems, we use some techniques similar to those from [5]. In addition, we prove a theorem concerning the chromatic numbers of arbitrary spaces $\mathbb{R}^n_K$ with segments of "forbidden distances". Namely, we define the value $\chi(\mathbb{R}^n_K, A)$ as the minimum number of colors which are needed to paint all the points in $\mathbb{R}^n_K$ so that any two points at any distance $x \in A$ apart receive different colors. Let $A' = [1, l]$.

Theorem 3. *The following five results hold.*

1. *One has* $\chi(\mathbb{R}^n_K, A') \leq (2(l+1)+o(1))^n$.
2. *Let* $p > 2$. *Then* $\chi(\mathbb{R}^n_p, A') \leq (2^{c_p}(l+1)+o(1))^n, c_p < 1, c_p \to 0$ *with* $p \to \infty$.
3. *Let* $l \geq 2$. *Then* $\chi(\mathbb{R}^n_K, A') \geq (l/2)^n$.
4. *Let* $l \geq 2$. *Then* $\chi(\mathbb{R}^n_p, A') \geq (b \cdot l)^n$, *where* $b = \frac{\sqrt[p']{2}}{2}$ *and* $p' = \max\left\{p, \frac{p}{p-1}\right\}$.
5. *Let* $l \geq 2$. *Then* $\chi(\mathbb{R}^n_2, A') \geq (b \cdot l)^n$, *where* $b \approx 0,755 \cdot \sqrt{2}$.

Theorem 3 has an interesting corollary. Indeed, it is known that

$$(c_1 m)^{c_2 n} \leq \max_{A,\ |A|=m} \chi(\mathbb{R}_2^n, A) \leq (3 + o(1))^{nm} \tag{1}$$

with some absolute constants $c_1, c_2 > 0$. The first result is done in [3]; the second one is an immediate consequence of a bound from [6]. Other results of this kind can be found in [7]. Anyway, in order to obtain a lower bound like in (1), one should use a set $A_0 = \left\{\sqrt{2p}, \ldots, \sqrt{2mp}\right\}$, where p is a certain prime number (see [7]). This is due to the specificity of the linear-algebraic method in combinatorics (see [3], [8]) — the only method which is known to provide good lower bounds for the chromatic numbers. However, it follows from assertion 1 of Theorem 3 that

$$\chi(\mathbb{R}_2^n, A_0) \leq \left(2\left(\sqrt{m}+1\right) + o(1)\right)^n,$$

since $A_0 \subset \left[\sqrt{2p}, \sqrt{2mp}\right]$ and so $l = \sqrt{m}$. In other words, Theorem 3 shows, in particular, that

$$\ln\left(\chi(\mathbb{R}_2^n, A_0)\right) = \Theta(n \ln m).$$

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On the number of biLipschitz classes of delone sets

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We study properties of biLipschitz equivalence relationship between discrete sets. The question about biLipscitz equivalence of Delone sets was raised by M.Gromov in paper [1].

Let M be a metric space. Denote by $B_\rho(x)$ a closed ball and by $B^\circ_\rho(x)$ an open ball of radius ρ with center at point x. A set $\mathcal{A} \subset M$ is a *Delone set* if for some real r and R such that $0 < r < R$ the following two conditions hold:

(1) $B^\circ_r(x) \bigcap B^\circ_r(y) = \varnothing$ for any $x, y \in \mathcal{A}$,

(2) $\bigcup\limits_{x \in \mathcal{A}} B_R(x) = M$,

Sometimes, Delone sets are called *separated nets*.

Delone sets $\mathcal{A} \subset M_1$ and $\mathcal{B} \subset M_2$ in possibly different metric spaces M_1 and M_2 are *biLipschitz equivalent*, if there exists a bijection $F : \mathcal{A} \to \mathcal{B}$ and a real number $\lambda \geq 1$ such that for every $x, y \in \mathcal{A}$ holds

$$\frac{1}{\lambda} d_{M_1}(x, y) \leq d_{M_2}(F(x), F(y)) \leq \lambda d_{M_1}(x, y).$$

If we mind the value of λ we call $\mathcal{A}$ and $\mathcal{B}$ λ-*bijective*. The mapping F for which the last inequality holds, is called λ-*biLipschitz*. If two sets are biLipschitz equivalent we say that they belong to the same biLipschitz class.

We mention some results on biLipschitz equivalence of Delone sets in Euclidean and non-Euclidean spaces. O. Bogopolsky [2] proved that any two Delone sets in hyperbolic space $\mathbb{H}^d$ are biLipschitz equivalent. P. Papasoglu [3] showed biLipschitz equivalence (as discrete metric spaces) of two homogeneous trees even with different valences $k \geqslant 3$ and $n \geqslant 3$.

In the case of Euclidean space $\mathbb{E}^d$ D. Burago and B. Kleiner (see [4]) and C. McMullen (see [5]) independently proved the existence of a Delone set which is not equivalent to the integer lattice $\mathbb{Z}^d$.

We generalize results of paper [4]. The main result obtained is the following

Theorem. *For every integer $d \geq 2$ the set of biLipschitz classes in $\mathbb{E}^d$ has cardinality continuum.*

All further arguments refer to the space $\mathbb{E}^d$.

The upper estimate for cardinality is trivial due to one A. Garber's lemma (see [6]). To establish the lower estimate we will construct a continuum family of pairwise non-equivalent Delone sets. All these sets will belong to some special class.

Consider a rectangular coordinate system in $\mathbb{E}^d$. Parallelepipeds (cubes) with edges parallel to coordinate lines are called *coordinate*.

Let Q be a coordinate cube. By $m(Q)$ denote its vertex with the least sum of coordinates.

Consider a tiling T of $\mathbb{E}^d$ into coordinate cubes such that edge length of every cube belongs to $[1, L]$. The set $\mathcal{A} = \{m(Q) : Q \in T\}$ is obviously a Delone set. We will call all sets obtained by such a construction *L-special*.

There are two crucial points in our proof.

The first one is lemma being a discrete analog of Burago and Kleiner's theorem about Jacobians (see [4, Theorem 1.2]). This lemma allows us for any λ to construct local obstacles in Delone sets for being λ-bijective to the integer net $\mathbb{Z}^d$.

The second point is considering a continuum set of pairwise non-confinal $(0,1)$-sequences. Each sequence will encode a Delone set, or, more precisely, an arrangement of constructed obstacles in it. Using the property of non-confinality we will prove that these sets are pairwise non-biLipschitz equivalent.

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К вопросу о перечислении архимедовых многогранников в пространстве Лобачевского

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Хорошо известна та связь, которая имеется между архимедовыми (трехмерными, конечными) многогранниками и соответствующими архимедовыми разбиениями двумерной сферы (см., напр, [1]). Полный перечень всех архимедовых (равноугольно полуправильных) многогранников был хорошо известен еще древним грекам. Аккуратное изложение изогональных (и изоэдральных) разбиений сферы (архимедовы разбиения являются лишь частью изогональных разбиений) читатель может найти, например, в работе Е.С.Федорова [1]. Основы теории изоэдральных разбиений плоскости Лобачевского фактически изложены в широко известном мемуаре А.Пуанкаре [2]. Подробное изложение (с точки зрения сортов) теории планигонов эвклидовой плоскости дано в работах Б.Н.Делоне и его учеников [3] и [4]. В данном сообщении предлагается подход к перечислению всех архимедовых (а, следовательно, и дуальных к ним) разбиений плоскости Лобачевского.

Условимся придерживаться общепринятого определения архимедова (равноугольно-полуправильного) многогранника как (выпуклого) многогранника, все грани которого - правильные многоугольники, среди граней которого имеются различные, а группа симметрии многогранника действует транзитивно на множестве его вершин. Разбиение (нормальное) двумерного пространства постоянной кривизны на правильные многоугольники называем архимедовым (коротко: -разбиением), если среди многоугольников есть различные, а группа симметрии разбиения действует транзитивно на множестве его вершин (узлов). Разбиение (нормальное) на равные правильные многоугольники договоримся называть платоновыми, ибо они приводят к платоновым (т.е. правильным) многогранникам. Аналогично определяются платоновы и -разбиения и архимедовы многогранники в более высоких размерностях.

В сообщении показывается, что, как и в случае правильных многогранников в и платоновых разбиений , в пространстве Лобачевского появляются архимедовы многогранники с бесконечным числом правильных конечных граней (и бесконечное число им соответствующих -разбиений). Показывается, что все те приемы, которые используются для получения архимедовых многогранников в случае эвклидова пространства , применимы и для получения архимедовых многогранников в пространстве Лобачевского (и архимедовых разбиений плоскости Лобачевского). Более того, в пространстве Лобачевского появляются новые типы архимедовых многогранников и новые счетные серии архимедовых многогранников. При этом появляющиеся новые серии все дальше и дальше качественно уходят от имеющихся сферических и эвклидовых аналогов. Это наводит на мысль о том, что таким способом всех серий не перебрать.

Выход из создавшегося положения подсказывает мемуар А.Пуанкаре [2], посвященный фактически теории планигонов на плоскости Лобачевского. Возникает естественная идея классификации -разбиений плоскости Лобачевского по родам поверхностей в стиле А.Пуанкаре. В сообщении показывается как эта идея может быть реализована практически. Из приведенных примеров становится ясно преимущество предлагаемого подхода.

В заключении сообщения будет обращено внимание на использование полученных результатов для построения новых разновидностей многогранников с правильными гранями в , равногранно-полуправильных многогранников, новых правильных и архимедовых многогранников с бесконечными гранями (см. [5], [6]).

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Local criterion for crystallographic tilings of Euclidean plane

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A tiling T of space by polyhedra is called crystallographic (more concretely, m-hedral) if the number of orbits of tiles w.r.to the symmetry group $Sym(T)$ is finite and equal to m . For simplicity, we consider just face-to-face (in Russian literature after Delone they are called normal) tilings. The definition of m-hedral tiling is based on a global concept of the symmetry group of a tiling. The local theory, started in works by Delone, Galiulin et al and developed in works by Dolbilin, Shtogrin, was aimed to give a groupless description of a crystallographic tiling in terms of congruent classes of coronae.

The corona $C_n(P)$ about a cell P of radius n is defined in a recurrent way. If $n = 0$ $C_0(P)$ is defined as the cell P itself. Let C_{n-1} be already determined then the corona $C_n(P)$ is defined as a polyhedral subcomplex in T consisting of $C_{n-1}(P)$ and all tiles of T sharing a common hyperface with tiles from this corona. Two coronae $C_k(P)$ and $C_k(P)$ belong to one class if there is an isometry of space which moves the corona $C_k(P)$ and $C_k(P)$ and the center P into the center The number of classes of coronae of radius k is denoted by N_k. For a given corona $C_k(P)$ denote by $G_k(P)$ a group of all symmetries of $C_k(P)$ leaving the center P invariant. Note that whereas N_k is monotonically non-decreasing function of k, the sequence of groups is monotonically reducing: $G_1 \ni G_2 \ni$

Local Theorem (Delone, Dolbilin, Galiulin, Shtogrin). Given a d-space (Euclidean or Spherical, or Lobachevski) and natural number m, a tiling T of space is m-hedral if and only if there is such a positive integer k that (1) $N_{k-1} = N_k = m$; (2) Groupes $G_{k-1}(P) = G_k(P)$ for any cell P in the T.

In particular, the local theorem implies an upper bound K for the radius of coronae such that if for a tiling $N_K = m$ then the tiling is m-hedral. However, this upper bound is very rough.

In the talk we will discuss recent results concerning reasonable estimates for the radius K of coronae such that the condition $N_K = m$ implies a tiling to be m-hedral tiling. The next theorem is principal:

Theorem (E.M.). Given m-hedral tiling in Euclidean plane, then:

(1) in any cell the number of its edges does not exceed $12m - 6$;

(2) The order of the group G_{2m-1} for all cells in T does not exceed 12.

Theorem. The m-hedrality criterion. A tiling in Euclidean plane is m-hedral if and only if $N_{5m} = m$;

Consistency on cubic lattices for determinants of arbitrary orders

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We consider relations on elementary $N \times N$ squares, $N > 2$, of the square lattice $\mathbb{Z}^2$, and propose a new type of consistency conditions on cubic lattices that is connected to bending elementary $N \times N$ squares, $N > 2$, in the cubic lattice $\mathbb{Z}^3$. For an arbitrary N we prove such consistency on cubic lattices for relations defined by the condition that determinants of values of the field at the points of the square lattice $\mathbb{Z}^2$ that are contained in elementary $N \times N$ squares vanish. We also consider some modifications and generalizations of this consistency principle.

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Unfoldings of doubly coverd polyhedra and space-fillers with minimum surface area[1]

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A *doubly covered square* is a degenerated polyhedron consisting of two congruent squares whose corresponding edges are identified. The geometric properties of convex unfoldings (developments) of a doubly covered square were studied by J. Akiyama and etc.[1], all such unfoldings were determined by them, and it was showed that such unfoldings are plane-fillers. We extend these results from the plane to the 3-space ([2]).

Definition 1. Let P be a polyhedron. The *doubly covered* P (denoted by $D(P)$) is the degenerated polytope in the 4-space consisting P and its congruent copy (denoted by P^*) whose corresponding faces are identified.

Definition 2. A body W (homeomorphic to a closed unit ball in $\mathbb{R}^3$) is called an *unfolding* of $D(P)$ for a polyhedron P if there is a continuous map (denoted by $f_{W,D(P)}$) from W onto $D(P)$ such that

(i) $f_{W,D(P)}$ is locally isometric on the interior of W, and

(ii) $f_{W,D(P)}$ has no 3-dimensional overlaps (that is, for disjoint sets of W the images have no common interior points).

A *parallelohedron* is defined as a polyhedron whose parallel copies tile the 3-space $\mathbb{R}^3$ in face-to-face manner. It was proved by E. S. Fedrov that the convex parallelohedra can be classified into five topological types: the cube, the hexagonal prism, the rhombic dodecahedron, the dodecahedron with eight rhombic and four hexagonal faces (the elongated dodecahedron), and the truncated octahedron.

Theorem 1. *All five types of parallelohedra can be obtained as unfoldings of doubly covered cuboids (rectangular parallelepipeds).*

By studying the geometric properties of convex unfoldings W of $D(P)$ for a cuboid P, under the assumption that W contains P, we can determine all such unfoldings. We define a *generalized rhombic dodecahedron*, a *generalized elongated dodecahedron* and a *generalized truncated octahedron.*

Theorem 2. *Convex unfoldings W of doubly covered cuboids $D(P)$ for a cuboid P are parallelepipeds, k-gonal right prisms ($3 \leq k \leq 6$), generalized rhombic dodecahedra, generalized elongated dodecahedra and generalized truncated octahedra, under the assumption that W contains P.*

A body W in the 3-space is called a *space-filler* of congruent copies of W tile the 3-space with no gaps and no 3-dimensional overlaps.

Theorem 3. *Every unfolding of $D(P)$ for a cuboid P is a space-filler.*

[1] This is a joint work with Jin-ichi Itoh

We can extend those results from cuboids to more general polyhedra called reflective space-fillers which are classified into seven types by H. M. Coxeter : the three types of tetrahedra, the three types of triangular right prisms and the cuboid ([2]).

As application of Theorem 2 and Theorem 3, we study the Kelvin's problem related to finding convex space-fillers with minimal surface area ([3]).

Theorem 4. *Among all convex unfoldings of the doubly covered cuboid with its edge lengths* $\sqrt{2}, \sqrt{2}$ *and one, the truncated octahedron has the minimum surface area.*

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The transition constant for arithmetic hyperbolic reflection groups

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The transition constant was introduced in our 1981 paper and denoted as N(14). This constant is fundamental since if the degree of the ground field of an arithmetic hyperbolic reflection group is greater than N(14), then the field comes from special plane reflection groups. In recent paper, we gave its upper bound 56. Using similar but more difficult considerations, here we show that the upper bound is 25.

As applications, we show that the degree of ground fields of arithmetic hyperbolic reflection groups in dimensions at least 6 has the upper bound 25 (it was 56 before); in dimensions 5, 4, and 3 it has the upper bound 44 (in our papers, it was 138, and 909 before).

These results and developed methods will be important for further classification of these groups. See details in arXiv:0910.5217.

The normal curvatures of hypersurfaces in Hilbert geometry[1]

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Consider a bounded open convex domain $U \subset \mathbb{R}^n$ such that its boundary is a C^∞ hypersurface with positive normal curvatures in $\mathbb{R}^n$ with Euclidean norm $\|\cdot\|$. For a point $x \in U$ and tangent vector $y \in T_xU = \mathbb{R}^n$ let x_- and x_+ – be the intersection points of the rays $x + \mathbb{R}_- y$ and $x + \mathbb{R}_+ y$ with *absolute* ∂U. Then the Hilbert metric is defined as follows: $F(x,y) = \frac{1}{2}\|y\|\left(\frac{1}{\|x-x_-\|} + \frac{1}{\|x-x_+\|}\right)$. Hilbert geometry is the generalization of the hyperbolic geometry in Klein interpretation (when $U = B^n_r$). Hilbert geometries are also Finsler spaces of constant negative flag curvature -1 .

The normal curvature of hypersurface in Finsler space is defined as follows [1]. Let $\varphi : N \to M^n$ be a hypersurface in Finsler manifold M^n. A vector $\mathbf{n} \in T_{\varphi(x)}M^n$ is called the normal vector to N at a point $x \in N$ if $\mathbf{g_n}(y, \mathbf{n}) = 0$ for all $y \in T_xN$. The *normal curvature* $\mathbf{k_n}$ at a point $x \in N$ in direction $y \in T_xN$ is defined as $\mathbf{k_n} = \mathbf{g_n}(\nabla_{\dot{c}(s)}\dot{c}(s)|_{s=0}, \mathbf{n})$, where $\dot{c}(0) = y$, and $c(s)$ is a geodesic in induced connection on N, $\mathbf{n}$ – chosen unit normal vector.

Hadamard proved that the compact orientable immersed in Euclidean space hypersurface with positive Gaussian curvature is embedded as the boundary of convex body [3].

C. Currier proved the following generalization of Hadamard theorem for immersions in the hyperbolic space L^n.

Theorem ([5]). *Let $\overline{M}$ be a complete connected C^∞-riemannian n-dimensional manifold, $n \geqslant 2$. Suppose that $f\colon \overline{M} \to L^n$ is a C^∞-isometric immersion of $\overline{M}$, and there exists a smooth normal vector field ν along f such that all the eigenvalues of the second form of the manifold $M = f(\overline{M})$ in L^n with respect to ν are greater of equal than 1. Let there exists a point $p \in M$ at which all the normal curvatures are strictly greater than 1. Then M is an embedded compact hypersurface which are diffeomophic to the sphere S^{n-1}.*

A.A. Borisenko ([4]) generalized Currier's theorem for immersions in a Hadamard manifold. He also obtained the extremal property of hyperbolic space.

We generalize Currier's theorem for Hilbert geometry.

Consider an immersion $\varphi : M \to U$ of C^∞-hypersurface M in n-dimensional Hilbert geometry. Denote by $\partial_\infty M$ the ideal boundary of M i.e. the intersection of all limit points of M with the absolute ∂U. We call the hypersurface M to be *regular up to the absolute* if each point $p \in \partial_\infty M$ has a neighborhood B such that the immersion $\varphi|_B : M \to U$ is extendable to the diffeomorphism $\bar{\varphi}|_B : \bar{M} \to \bar{U}$.

Theorem. *Consider n-dimensional Hilbert geometry based on the domain $U \in \mathbb{R}^2$, which is a bounded open set with the boundary C^∞-hypersurface with positive normal curvatures. Consider C^∞-immersion $\varphi : M \to U$ of a complete connected regular up to the absolute hypersurface M in U. Let all the normal curvatures of M satisfies $\mathbf{k_n} \geqslant k_0 > 1$. Then M is an embedded compact hypersurface which are diffeomophic to the sphere S^{n-1}.*

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Singularities of saddle spheres

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Segre's theorem asserts the following: *let a smooth closed simple curve $c \subset S^2$ have a non-empty intersection with any closed hemisphere. Then c has at least 4 inflection points.*

In the talk, we go one dimension higher: we replace S^2 by S^3. Instead of simple curves, we treat immersed piecewise linear saddle surfaces which are homeomorphic to S^2 ("saddle spheres"). We prove that a piecewise linear saddle sphere $\Gamma \subset S^3$ necessarily has *inflection* or *reflex faces.* The latter replace inflection points and should be considered as singular phenomena.

This object is not chosen just by chance: the study of closed saddle surfaces was originally motivated by A.D. Alexandrov's problem.

As an application, we prove that a piecewise linear saddle surface cannot be altered in a neighborhood of its vertex maintaining its saddle property.

On axiomatic parametrization

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When Euclidean geometry was the only considered one, nobody cares how it relays with other geometric systems. However, when non–Euclidean geometries were developed, their axiomatics become important. Construction of new geometry is not trivial in both synthetic and analytic ways. Felix Klein in [1] proposed Erlangen Program, aimed to classify and characterize geometries on the basis of projective geometry and group theory. In his work [2], Isaak Yaglom says that: "finding a general description of all geometric systems [was] considered by mathematicians the central question of the day".

Interestingly, two great contributors in construction of hyperbolic geometry, János Bolyai and Nikolai Lobachevsky adopted different ways on achieving their goal. While Bolyai dropped the V-th postulate of Euclid and developed 'absolute geometry', Lobachevsky changed it in a certain way. Author developed a concept of axiomatic depending on parameters, a single set of primitives and axioms depending on some parameters that can describe any homogeneous geometry[1], as well as an uniform model for all homogeneous spaces, depending on same parameters. Author's project, *GeomSpace* [4], is based on this uniform model.

The advantages of axiomatic parametrization are, among others:

- Elaboration of common terminology among different geometries.
- Classification of homogeneous spaces.
- Comparison of geometric properties of different geometries.
- Construction and study of new geometries with given properties.
- Possibility to formulate theorem depending on parameters that are valid for all geometries, and demonstrate them parametric, once for all geometries.
- Develop of depending on parameters equations, equally valid for all geometries, and deduce them once for all geometries.

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Invariant polyhedra for families of linear operators

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The notion of the joint spectral radius (JSR) of several operators appeared in early 60th in a short work of J.K.Rota and G.Strang. Now it has found numerous applications in the control theory, functional analysis, approximation theory, number theory, wavelets, coding theory etc. The joint spectral radius of a family $\mathcal{M} = \{A_1, \dots, A_m\}$ of linear operators acting in $\mathbb{R}^d$ is defined as

$$\hat{\rho}(\mathcal{M}) \quad = \quad \lim_{k\to\infty} \quad \max_{d_1,\dots,d_k \in \{1,\dots,m\}} \left\| A_{d_1} \cdots A_{d_k} \right\|^{1/k} .$$

So, JSR is the exponent of the maximal growth of products of those operators. For example, if the family $\mathcal{M}$ is irreducible (i.e., the operators $A_1, \dots, A_m$ do not have a nontrivial common invariant subspace), then $\max_{d_1,\dots,d_k} \|A_{d_1} \cdots A_{d_k}\| \asymp \lambda^k$ where $\lambda = \hat{\rho}(\mathcal{M})$. This, in particular, implies the crucial property of JSR: $\rho(\mathcal{M}) < 1$ if and only if there is a norm in $\mathbb{R}^d$, in which all the operators $A_1, \dots, A_m$ are contractions.

One the main problems in the study of JSR is its computation or estimation for given operators. This problem is known to be NP-hard. There are no algorithms polynomial with respect to both the dimension d and the accuracy ε of approximation.

We describe a geometrical approach using the notions of extremal norms and invariant convex bodies of linear operators. A convex body $K \subset \mathbb{R}^d$ is *invariant* for a family $\mathcal{M}$, if

$$A_i K \ \subset \ \lambda K \,, \qquad i = 1, \dots, m \,,$$

where $\lambda = \hat{\rho}(\mathcal{M})$. Invariant bodies exist for any irreducible family, and may not be unique. It appears that in most of practical cases the invariant body is a polyhedron, and can be efficiently found. This leads to exact computation of JSR.

We analyze the structure of invariant polyhedra and methods for their construction. We also present several applications to problems of real analysis, combinatorics, and number theory, where constructing invariant polyhedra gave complete solutions.

Borsuk and Nelson – Hadwiger problems for spheres[1]

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This work is dealt with two classical and closely connected problems of combinatorial geometry. The first problem was proposed in 1933 by K. Borsuk who asked whether any set of diameter 1 in $\mathbb{R}^d$ can be divided into $d+1$ parts of smaller diameter (see [1]). The second problem is due to E. Nelson and H. Hadwiger. Initially, it was in finding the value $\chi(\mathbb{R}^d)$ equal to the minimum number of colors which are needed to paint all the points in $\mathbb{R}^d$ so that any two points at distance 1 apart receive different colors (see [2]).

The history of Borsuk's question as well as that of the Nelson – Hadwiger problem is very interesting and even somehow dramatic. It can be found in many books and surveys (see, e.g., [3], [4], [5],[6]).

Important variants of both problems have been proposed for spheres in $\mathbb{R}^d$. Let $S_r^{d-1} \subset \mathbb{R}^d$ be the sphere of radius r with center at the origin. Denote by $f_r(d)$ the minimum number of parts of smaller diameter, into which an arbitrary set $\Omega \subset S_r^{d-1}$ of diameter 1 can be decomposed. Also, let $\chi(S_r^{d-1})$ be the minimum number of colors needed to paint all the points of the sphere so that any two points at distance 1 apart receive different colors.

Of course, we have $r \geq 1/2$. Moreover, $\chi(S_{1/2}^{d-1}) = 2$, $f_{1/2}(d) = d+1$. The last result is essentially equivalent to the classical Borsuk – Ulam theorem in topology.

For $r > 1/2$, the value $\chi(S_r^{d-1})$ was studied, in particular, by L. Lovász in [7]. The exact assertion of Lovász is as follows: *for any $r > \frac{1}{2}$ and $d \in \mathbb{N}$, the inequality holds $\chi(S_r^{d-1}) \geq d$; if $r < \sqrt{\frac{d}{2d+2}} \sim \frac{1}{\sqrt{2}}$, i.e., the length of any side of a regular d-simplex inscribed into S_r^{d-1} is smaller than 1, then $\chi(S_r^{d-1}) \leq d+1$.* Although this result is widely cited (see, e.g., [8]), its second part is completely wrong. Actually, for every $r > \frac{1}{2}$, the quantity $\chi(S_r^{n-1})$ grows exponentially, not linearly. Our results are given below.

Theorem 1. *For any $r > \frac{1}{2}$, there exist a constant $\gamma = \gamma(r) > 1$ and a function $\varphi(d) = \varphi(d,r) = o(1)$, $d \to \infty$, such that for every $d \in \mathbb{N}$, the inequality holds*

$$\chi(S_r^{d-1}) \geq (\gamma + \varphi(d))^d.$$

Theorem 2. *There exists a constant $c > 0$ such that for any sequence of radii r_d satisfying the inequality*

$$r_d \geq \frac{1}{2} + \frac{c}{d^{0.475}},$$

we have the bound

$$\chi(S_{r_d}^{d-1}) > d+1, \quad \forall\ d \geq d_0.$$

Theorem 3. *There exists a constant $c > 0$ such that for any sequence of radii r_d satisfying the inequality*

$$r_d \leq \frac{1}{2} + \frac{c}{d},$$

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we have the bound

$$\chi(S_{r_d}^{d-1}) \leq d+1, \quad \forall\ d \geq d_0.$$

As for Borsuk's problem, it follows from a paper by J. Kahn and G. Kalai (see [4]) that $f_r(d)$ grows like $c^{\sqrt{d}}$, $c > 1$, provided $r \sim \frac{1}{\sqrt{2}}$ (Borsuk's question has a negative answer). However, for other values of r, no one knew how to produce estimates. We succeded in finding non-trivial bounds for $f_r(d)$ with any $r > 1/2$.

Theorem 4. *For any $r > \frac{1}{2}$, there exist numbers $k = k(r) \in \mathbb{N}$, $c = c(r) > 1$ and a function $\delta = \delta(d) = o(1)$ such that*

$$f_r(d) \geq (c+\delta)^{\sqrt[2k]{d}}.$$

Theorem 5. *Let $r = r(d) = \frac{1}{2} + \varphi(d)$, where $\varphi = o(1)$ and $\varphi(d) \geq c\frac{\ln\ln d}{\ln d}$ for all d and a large enough $c > 0$. Then, there exists a d_0 such that for $d \geq d_0$, $f_{r(d)}(d) > d+1$.*

At the same time, we get

Theorem 6. *Let $r = r(d) = \frac{1}{2} + \varphi(d)$, where $\varphi = O(1/d)$. Then, $f_r(d) \leq d+1$.*

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Delone Sets and the Homometry Problem

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Two Delone sets A and B in E^n are said to be *homometric* if $A - A = B - B$. 75 years after the phenomenon was first noted (in crystallography) many questions remain open. Which sets A have homometric partners, and how many? How can we find them? What is the geometry of homometric pairs? I will explain what is known and what is not known about the homometry problem, beginning with Delone's brilliant reconstruction of mathematical crystallography from (r,R) systems in the 1930s and concluding with conundrums in quasicrystals.

Tiling by rectangles and discrete inverse problems

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This is a joined work with M. Prasolov [6].

Tiling problems are popular because they are very visual but often hard to solve [3,4,5]. They have applications in architecture and design. The interest to these problems always grows because of discovery of their relationship with discrete harmonic and complex analysis, probability theory, physics of networks [1,5].

We solve the following problem in certain particular cases: *which polygons can be tiled by rectangles of given shapes*? A classical case of rectangles tilable by squares was considered by M. Dehn in 1903. The general problem for ”signed” tilings was solved by K. Keating and J. King [4].

Tilings by rectangles have a celebrated physical interpretation with direct-current circuits, found by R.L. Brooks, C.A.B. Smith, A.H. Stone and W.T. Tutte. Our new approach is based on application of inverse problems for direct- and alternating-current circuits.

Our first result is a necessary condition for a rectangle to be tilable by rectangles of given ratios. By the *ratio* of a rectangle we mean the horizontal side divided by the vertical one.

Theorem 1. *Suppose that a rectangle of ratio c can be tiled by rectangles of ratios $c_1, \dots, c_n$. Then $c = C(c_1, \dots, c_n)$ for some rational function $C(z_1, \dots, z_n)$ such that*

- $C(z_1, \dots, z_n)$ has rational coefficients, i.e., $C(z_1, \dots, z_n) \in \mathbb{Q}(z_1, \dots, z_n)$;
- $C(z_1, \dots, z_n)$ is degree 1 homogeneous, i.e., $C(tz_1, \dots, tz_n) = tC(z_1, \dots, z_n)$;
- if $Re\, z_1, \dots, Re\, z_n > 0$ then $Re\, C(z_1, \dots, z_n) > 0$.

Case $n = 1$ (respectively, $n = 2$) of both Theorem 1 and its converse was proved by M. Dehn (respectively, by C. Freiling, M. Laczkovich and D. Rinne). We dot know whether the converse theorem is true for $n \geq 3$.

Our second result is a criterion for a rectangle to be tilable by rectangles similar to it but not all homothetic to it.

Theorem 2 *For a number $c > 0$ the following* 3 *conditions are equivalent*:

- a rectangle of ratio c can be tiled by rectangles of ratios c and $1/c$ (in such a way that there is at least one rectangle of ratio $1/c$ in the tiling);
- the number c^2 is algebraic and all its algebraic conjugates distinct from c^2 are negative real numbers.
- for certain positive rational numbers $d_1, \dots, d_m$ we have

$$\cfrac{1}{d_1 c + \cfrac{1}{d_2 c + \dots + \cfrac{1}{d_m c}}} = c.$$

This result is analogous to a description of rectangles whose similar copies tile a square, obtained by C. Freiling, M. Laczkovich, D. Rinne and G. Szekeres [3]. A short physical proof of the latter result is also obtained. The proof uses alternating-current circuits and reduces the result to a simple inverse problem for them solved by R. Foster and W. Cauer in 1920s.

Our third result is a criterion for a (not necessarily convex) polygon to be tilable by squares. Such a criterion for a rectangle is given by the M. Dehn theorem, and for an L-shaped hexagon was obtained by R. Kenyon [3]. We reduce the general problem to an inverse problem for dirrect-current electrical circuits solved recently by Y. Colin de Verdiere, E. Curtis and J. Morrow [2].

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Quasiperiodic tilings and cubic irrationalities[1].

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Let $\beta > 1$ be a cubic Pisot unit. Then for any positive real x we can obtain the greedy expansion

$$x = \sum_{i=N_0}^{\infty} a_{-i}\beta^{-i} \tag{5}$$

with $a_i \in \mathbb{Z}$ and

$$|x - \sum_{i=N_0}^{N} a_{-i}\beta^{-i}| < \beta^{-N} \tag{6}$$

Let Φ be a map from $\mathbb{Q}(\beta)$ to $\mathbb{R}^2$ defined by $\Phi(x) = (\operatorname{Re} x^{(1)}, \operatorname{Im} x^{(1)})$ if algebraic conjugates to β are complex, and $\Phi(x) = (x^{(1)}, x^{(2)})$ if algebraic conjugates to β are real. Here $*^{(i)}$ are conjugations in the field $\mathbb{Q}(\beta)$. Now suppose that w runs all possible finite fractional parts with (6) and S_w is the set of the sums (5) whose fractional parts coincides with w. Akiyama [1] proved that if the expansion (5) is finite for any $x \in \mathbb{Z}[\beta^{-1}]$ then we have a self-affine quasiperiodic plane tiling $Til(\beta)$

$$\mathbb{R}^2 = \coprod_{w} \overline{\Phi(S_w)}. \tag{7}$$

He also establish some interesting results about the connection of the algebraic properties of β and geometric properties of $Til(\beta)$. Earlier Rauzy studied special case of this tiling in the case $\beta^3 - \beta^2 - \beta - 1 = 0$ [2].

In [3] we prove some new geometric properties of the tiling $Til(\beta)$.

Consider a similarity transformation which maps the tile with the point of origin to some fixed tile T from $Til(\beta)$. The image of the point of origin under this transformation is called a Rauzy point of the tile T. Denote by $R(\beta)$ the set of all Rauzy points from the tiling. Note that the tiling $Til(\beta)$ consists of only finite types of tiles. Let $R^{(i)}(\beta)$ be the set of all Rauzy points of the tiles of type i.

Theorem 1. *The set $\Phi^{-1}(R^{(i)}(\beta))$ is an intersection of the ring $\mathbb{Z}[\beta^{-1}]$ with some right-open interval. Moreover, $\Phi^{-1}(R^{(i)}(\beta)) \subseteq [0;1)$.*

Two tiles from $Til(\beta)$ are neighbouring if they have a common part of boundary. Let x be a Rauzy point of some tile T. The local star $S(x)$ is a set of vectors traced from this Rauzy point x to Rauzy points of tiles neighboring T. Each vector has weight, the number equal to the type of the neighboring tile.

Theorem 2. *The tiling $Til(\beta)$ has only finite type of local stars. Moreover, if $\widehat{R}^{(i)}(\beta)$ is the set of all Rauzy points with local star of type i then $\Phi^{-1}(\widehat{R}^{(i)}(\beta))$ is an intersection of the ring $\mathbb{Z}[\beta^{-1}]$ with some right-open interval.*

Now let $C_n(X)$ be the n-crown of some set of tiles X. Let $N_{Til(\beta)}(n)$ be a number of equivalence classes of n-crown of the tiles from $Til(\beta)$. The function $N_{Til(\beta)}(n)$ is called a complexity function of the tiling.

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This is a joint work with A.V.Maleev and V.G.Zhuravlev

Theorem 3. *In any tiling $Til(\beta)$ there exists the set $Nucl(\beta)$ such that the complexity function $N_{Til(\beta)}(n)$ is equal to the number of the tiles in n-crown $C_n(Nucl(\beta))$. Moreover, different tiles from $C_n(Nucl(\beta))$ have different n-crowns.*

Now let T_0 be a tile from $Til(\beta)$ consisting the point of origin. Then we have the folowing conjecture.

Conjecture 1. *For any tiling $Til(\beta)$ there exists a convex centrosymmetric polygon $pol(\beta)$ such that*

$$\lim_{n\to\infty} \frac{C_n(T_0) \setminus C_{n-1}(T_0)}{n} = pol(\beta).$$

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Surfaces in three-dimensional Lie groups

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We give a survey of recent progress in theory of surfaces in three-dimensional Lie groups and, in particular, expose results obtained via the spinor representation of surfaces.

The strong thirteen spheres problem[1]

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The Tammes problem

If n unit spheres kiss the unit sphere in $\mathbb{R}^d$, then the set of kissing points is an arrangement on the central sphere such that the (Euclidean) distance between any two points is at least 1. So the kissing number problem can be stated in other way: How many points can be placed on the surface of $\mathbb{S}^{d-1}$ so that the angular separation between any two points is at least 60^0?

Denote by d_N the largest angular separation that can be attained in a spherical code on $\mathbf{S}^2$ containing N points. In other words, how are N congruent, not overlapping circles on the sphere to distribute when their common radius of the circles has to be as large as possible? This question, also known as problem of the "enimated dictators", was first asked by the Dutch biologist Tammes (1930) who was led to this problem by examining the distribution of the openings on the pollen grains of different flowers.

The Tammes problem presently solved only for some small values of N: for $N = 3, 4, 6, 12$ by L. Fejes Tóth; for $N = 5, 7, 8, 9$ by Schütte and van der Waerden; for $N = 10, 11$ by Danzer; and for $N = 24$ by Robinson.

The Tammes problem for $N = 13$

The first unsolved case of the Tammes problem is $N = 13$ which is particularly interesting because of its relation to the kissing problem and the Kepler conjecture.

It's clear that the equality $k(13) = 12$ implies $d_{13} < 60^0$. Bóroczky and Szabo proved that $d_{13} < 58.7^0$. Recently Bachoc and Vallentin have shown that $d_{13} < 58.5^0$.

We note that one can construct an arrangement of 13 points on $\mathbf{S}^2$ such that the distance between any two points of the arrangement is at least 57.1367^0. This arrangement is shown in Fig. 1. In the paper we show that this arrangement is the best possible and so $d_{13} \approx 57.1367^0$

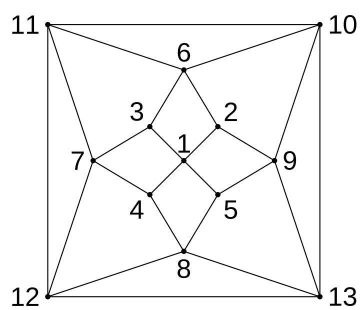


Рис. 1: Graph with best known d_{13}.

Irreducible graphs

Cosider some arrangement M of n points on the shpere. Mark all minimal distances between points. Denote this graph by $F(M)$.

Definition 6. *Irreducible graph* The graph $F(N)$ is *irreducible* if altering of each point of M does not improve minimal distance.

Irreducible graph has some properties (citation ??):

[1]This is a joint work with Oleg R. Musin.

1. Index of any vertex can be $0, 3, 4, 5$.

2. All angles of adjacent edges are less than π.

3. For $n \leq 17$, there are no possible septagons and higher.

4. For $n \leq 17$ free point can be inside ony hexagon, and only one point per hexagon.

5. Danzers trick. Consider some face f and its vertex v_i. Let v_i' be a point inside f and symmetrical to v_i above line $v_{i-1}v_{i+1}$ passing through adjacent vertices. So for some j distance $v_i'v_j$ should be less than minimal. Danzer proved that distance from v_i' to other vertices is grater than minimal distance. So if also each $v_i'v_j \leq d_13$ we can alter v_i to v_i'.

   Danzer's trick does not improve miniaml distance, but eliminates at least one edge of graph or splits face into smaller.

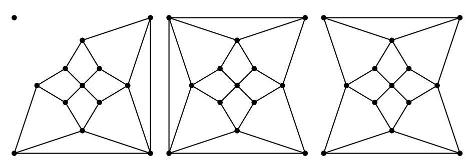
   (is it correct place ?? Danzer trick also doesnot improve minimal distance bbut decreases number of minimal distances of graph)

Scheme of the algorythm

We assume that best graph is known aggangement (fig. 1). We consider all planar graph satisfying properties of irreducible graphs: three-connected planar, index of each vertex is not more than 5, number of sides of each face is not more than 6.

We consider each such graph as possible irreducible with maximal minimal distance and trying to disproof that it is possible.

For checking feasability of a given graph we consider each angle of any graph face as independent variable. We write known constrains for these variables and later prove that this set of non-linear constrains does not have any solution. In the case of success we consider graph as unfeasable and eliminate it.



$\Gamma_{13}^{(3)}$ $\Gamma_{13}^{(1)}$ $\Gamma_{13}^{(2)}$

Рис. 2: Irreducible graphs infinitisimally close to G_{best}: with pentagon, with empty hexagon, with hexagon containing free point

After using program only three extra graphs are left (fig. 2). It is best possible variant for computer program, because for this graphs maximal minimal distance can be arbitrary close to assuming value of d_{13} – 57.1367^0.

Self-affine polyhedra and p-radius of linear operators

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Body $X \subset \mathbb{R}^d$ (convex compact with non-empty interior) is called *self-affine* with finite collection of nondegenerate affine operators of partitioning $A_1, \dots, A_k$ if $X = \bigcup_{i=1}^{k} A_k X$ and if $i \neq j$, then sets $A_i X, A_j X$ has no common inside points. Bodies $A_i X$ are elements of partition. Bodies $A_i A_j X, j = 1, \dots, k$ impose self-affine partition of body $A_i X, i = 1, \dots, k$, thus we can iterate given partition. Body X is called *segmenting* by collection of affine operators of partition $A_1, \dots A_k$, if $\mu_n(\varepsilon) = \mu(\cup A_{i_1} \dots A_{i_n} X | \ \mathrm{diam}(A_{i_1} \dots A_{i_n} X) > \varepsilon) \to 0$ by $n \to \infty$ for each ε.

Theorem 1. *Any segmenting self-affine body is a polyhedron.*

Thus, study of segmenting bodies reduces to study of polyhedrons. It makes sense to generalize the definition of segmenting polyhedra in the event of not self-affine polyhedra. Lat we have a body X and collection of affine operators (possibly degenerate) $A_1, \dots A_k$, such that $A_i X \subset X, i = 1, \dots k$. Body X is called *compressible* if for each $\varepsilon > 0$ there is a composition of source operators $\hat{A}_\varepsilon$, such that $\mathrm{diam}(\hat{A}_\varepsilon X) < \varepsilon$, or $\|\hat{A}_\varepsilon\| < \varepsilon$.

Theorem 2. *Polyhedron is not compressible if and only if there is a set G of nonintersecting faces, such that under action of each operator A_i, $A_i G \subset G$, in addition each face from G contains image of some face from G and $|G| \geqslant 2$.*

As a corollary we deduce a simple criterion to check, is a polyhedron segmenting.

The results about self-affine segmentic polyhedra can be applied in the theory of self-affine fractals. In [1] by finite partition of segment were built fractal curves in $\mathbb{R}^d$. It's possible to generalize ideas from this work to fractal surfaces and beneficate their combinatorics by self-affine segmentic polyhedrons. It should be noted, that it's not a constructive algorithm to find a fractal surface.

Let $K \subset \mathbb{R}^{d'}$ be fixed self-affine segmentic polyhedron with operators of partition $A_1, \dots, A_k$. Suppose we are given a family of affine operators $\widetilde{\mathcal{B}} = \{\widetilde{B}_1, \dots, \widetilde{B}_k\}$, acting in $\mathbb{R}^d$. Let $\mathcal{B} = \{B_1, \dots, B_k\}$ be the family of the associated linear operators in $\mathbb{R}^d$

A fractal surface of a family of affine operators $\widetilde{\mathcal{B}}$ is a summable function $v \in L_p(K) : K \to \mathbb{R}^d$ satisfying the equation:

$$v(t) = \widetilde{B}_m v(A_m^{-1}(t)), \ t \in A_m K, \ m = 1, \dots k \tag{8}$$

For a given $n \in \mathbb{N}$ and for any sequence $\sigma \in \{1, \dots, k\}^n$ we write Π_σ for the product $B_{\sigma(1)} \dots B_{\sigma(n)}$. Also for any $p \in [1, \infty)$ denote by $\mathcal{F}_n(p) = \mathcal{F}_n(p, \mathcal{B})$ the value $[k^{-n} \sum_\sigma \|\Pi_\sigma\|^p]^{1/p}$. For given $p \in [1, +\infty]$ the *p-radius of linear operators* of the family $\mathcal{B}$ is the value $\rho_p = \rho_p(\mathcal{B}) = \lim_{n \to \infty} [\mathcal{F}_n(p, \mathcal{B})]^{1/n}$.

Theorem 3. *For an irreducible family of affine operators Eq.* (8) *possesses a summable solution $v(t)$ if and only if $\rho_1(kr\mathcal{B}) < 1$. This solution is unique. If for some $p \in [1, +\infty]$ one has $\rho_p((kr)^{1/p}\mathcal{B}) < 1$, then $v \in L_p$. For $p < \infty$ the converse is also true: if $v \in L_p$, then $\rho_p < 1$. If $v \in L_\infty$, then $\rho_\infty \leqslant 1$.*

Results about compressible polyhedra are applied in the matrix theory. Suppose there is a collection of stochastic nonnegative matrixes $A_1, \dots, A_k$, acting in $\mathbb{R}^d$. Was found a simple criterion, that practically each product $A_{i_1} \dots A_{i_n} \dots$ converges. Herewith image of space converges to a line and composition converges to a matrix rank 1. In terms of Markov's chains it means converge of any Markov's process with probability 1.

Theorem 4. *For family of stochastic matrixes $A_1, \dots, A_k$ we have $\rho_p(A_1, \dots, A_k) < 1$ for all $p \geqslant 1$ if and only if for each two indexes $t_1, t_2 \leqslant d$ there is product Π of given matrixes such that for some t we have $(\Pi)_{ti} > 0$ and $(\Pi)_{tj} > 0$.*

In literature (see [2, 3, 4] and references therein) conditions, that for each sequence $\{i_t\}$ and collection of stochastic matrixes $A_1, \dots, A_k$, limit $\lim_{n\to\infty} \|A_{i_1}\mid_{\mathrm{span}\{K\}} \dots A_{i_n}\mid_{\mathrm{span}\{K\}}\| = 0$, were detailed studied. Such matrixes associate with convergence of subdivision-algorithms. In [2] it's shown that this check is not solvable algorithmic in a polynomial time. We weaken condition in suggest that this limit converges with probability 1 and represent polynomial algorithm to check.

In [2] was found the algorithm to check, is there fractal curve for family of stochastic matrixes. We represent algorithm, built with theorem 4, considerably simplifying check, moreover making generalization in case of fractal surfaces.

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On the study of dihedral folding tilings of the sphere

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A tiling of the sphere with disks is called dihedral if any disk of the tiling is congruent to one of two fixed disks. A dihedral edge-to-edge tiling of the sphere with geodesic polygons is called folding if all vertices are of even valency and the sums of alternating angles around each vertex are equal to π.

Portuguese mathematicians d'Azevedo Breda and Santos have found dihedral folding tilings of the sphere with spherical triangles and spherical parallelograms. In [1] these authors classified the symmetry groups of the obtained dihedral folding tilings of the sphere, as well as determined, for each case, the number of transitivity classes of polygons (isohedrality) and vertices (isogonality).

The idea of B. N. Delone to classify tilings using so-called Delone classes gave rise to some fruitful methods for obtaining tilings.

In works [2, 3] the author of this thesis classified 2-isohedral tilings of the sphere using Delone classes. It prompts another approach to researching dihedral folding tilings. First select 2-isohedral tilings with all vertices of even valency. It was found that tilings having this property are with two classes of triangles, with triangles and quadrangles, with triangles and pentagons, with triangles and hexagons. Then, examining the metric of tiling, check the sum of alternating angles around each vertex.

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Translated directions on the surface of the conformal space

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Consider a multi-dimensional surface $V_m \subset C_n$, referred to semi-isotropic semi-orthogonal frame $R = \{A_\lambda\}$, $\lambda, \mu = \overline{1, n+1}$. In this frame the equations: $\omega_0^\alpha = 0, \omega_\alpha^j = \Lambda_{\alpha k}^j \omega_0^k, \omega_i^\alpha = \Lambda_{ij}^\alpha \omega_0^j, \Lambda_{[ij]}^\alpha = 0$ $(i, j = \overline{1, m}, \alpha, \beta = \overline{m+1, n})$ are true. We are given the normal framing [1] of surface V_m, determined by the field of quasitensor x_i^0:$dx_i^0 + x_i^0 \omega_0^0 - x_j^0 \omega_i^j + \omega_i^0 = x_{ij}^0 \omega_0^j$. In this case, the normal connection $\nabla^\perp$ is induced on the surface $V_m \subset C_n$.

Presetting the multi-dimensional surface V_m in the conformal space C_n induces a regular m-dimensional quadratic hyperband H_m in projective space P_{n+1}. Consider a regular quadratic hyperband $H_m \subset \mathrm{P}_{n+1}$, that is mutual and dual way normalized by fields of normals of the first N_{n-m+1} and second N_{m-1} kinds.

The condition of parallelism of a smooth field of the directions $[\mathrm{A}_0\mathrm{M}]$, belonging to the field N_{n-m+1} of normals of the first kind of the hyperband $H_m \subset \mathrm{P}_{n+1}$ in normal connection $\nabla^\perp$, by a displacement along any curve due to the surface $\widetilde{V_m} \subset Q_n^2 \subset \mathrm{P}_{n+1}$, has the form:

$$dx^\alpha + x^\beta(2\omega_\beta^\alpha - \Theta_\beta^\alpha) + x^{n+1}(a_{n+1k}^\alpha \omega_0^k - a_{n+1}^\alpha x_s^0 \omega_0^s - g^{sj} x_j^0 \omega_s^\alpha) = x^\alpha \Theta, dx^{n+1} = x^{n+1}\Theta.$$

Based on these conditions is proved:

Theorem 1. *In any normal framing of the surface $V_{n-2} \subset C_n$, the field of 2-dimensional characteristics $[\mathrm{A}_0\mathrm{A}_{n-1}\mathrm{A}_n]$ of the hyperband $H_{n-2} \subset P_{n+1}$ translates in the normal connection $\nabla^\perp$.*

The characteristic $[\mathrm{A}_0\mathrm{A}_{n-1}\mathrm{A}_n]$ of the hyperband $H_{n-2} \subset \mathrm{P}_{n+1}$ by Darboux mapping is an image of 2-parametric bundle of hyperspheres $Q = \eta^\alpha A_\alpha + \eta^0 A_0$, tangential to each other at the point $A_0 \in V_{n-2}$. Theorem 1 can be formulated in terms of conformal space C_n:

Theorem 2. *In any normal framing of the surface $V_{n-2} \subset C_n$ field of 2-parameter bundle of tangential hyperspheres $Q = \eta^\alpha A_\alpha + \eta^0 A_0$ $(\alpha = n-1, n)$ of submanifold V_{n-2} translates in the normal connection $\nabla^\perp$.*

Let the field of lines $[\mathrm{A}_0\mathrm{M}]$ coincides with the field of invariant straight lines $h \equiv [\mathrm{A}_0\mathrm{N}_{n+1}]$. Then valid:

Theorem 3. *The field of invariant straight lines $h \equiv [\mathrm{A}_0\mathrm{N}_{n+1}]$ on the hyperband $H_m \subset P_{n+1}$, determined by a quasitensor field x_i^0, is parallel in the normal connection $\nabla^\perp$ if and only if the tensor A_{n+1k}^α is equal to zero.*

A line $[\mathrm{A}_0\mathrm{N}_{n+1}]$ in P_{n+1} by Darboux mapping is an image of the bundle of the orthogonal hyperspheres $P = \xi^{n+1}N_{n+1} + \xi^0 A_0$. Theorem 3 can be formulated in terms of the conformal space C_n:

Theorem 4. *Field of invariant bundle of hyperspheres $P = \xi^{n+1}N_{n+1} + \xi^0 A_0$ tangential to each other at the points $A_0 \in V_m$, determined by the field of quasitensor x_i^0, is parallel in the normal connection $\nabla^\perp$ if and only if the tensor A_{n+1k}^α is equal to zero.*

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Geometric approach to stable homotopy groups of spheres: abelian and quaternionic structure for mappings with singularities.

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Denote by $\mathbb{RP}^{n-d}$ the standard $n-d$–dimensional projective space, assuming that $n-d \equiv 1 \pmod 2$, denote by $S^{n-2k}/\mathbf{i}$ the standard $n-2k$–dimensional lens space $\pmod 4$, assuming $n-2k \equiv 3 \pmod 4$. Consider generic PL–mapping $d : \mathbb{RP}^{n-d} \to \mathbb{R}^n$ and denote by $(N^{n-2d}, \partial N)$ the polyhedron with boundary of self-intersection points of the mapping d. The this polyhedron $N^{n-2d} \setminus \partial N$ is defined by the formula:

$$Cl\{[(x,y)] \in \mathbb{RP}^{n-d} \times \mathbb{RP}^{n-d}/\sim \quad | \quad x \neq y, d(x) = d(y)\}.$$

The boundary ∂N of this polyhedron N^{n-2d} consists of critical points of the mapping d such that the following natural inclusion $\partial N \subset \mathbb{RP}^{n-d}$ is well-defined.

Analogically, consider generic PL–mapping $c : S^{n-2k}/\mathbf{i} \to \mathbb{R}^n$ and denote by $(L^{n-4k}, \partial L)$ the polyhedron with boundary of self-intersection points of the mapping c. Its boundary ∂L consists of critical points of the mapping c such that the following natural inclusion $\partial L \subset S^{n-2k}/\mathbf{i}$ is well-defined. Denote by $\bar{L}^{n-4k}$ the canonical 2-sheeted covering over the polyhedron L^{n-4k} (with ramification over the boundary ∂N). This covering is defined by the formula:

$$Cl\{(x,y) \in S^{n-2k}/\mathbf{i} \times S^{n-2k}/\mathbf{i} \quad | \quad x \neq y, c(x) = c(y)\}.$$

The following natural inclusion $\partial L \subset \bar{L}^{n-4k}$ and the following natural mapping $\bar{L}^{n-4k} \subset S^{n-2k}/\mathbf{i}$ (this mapping is an inclusion in the case $6k > n$) are well-defined.

Theorem 1. There exists a mapping $d : \mathbb{RP}^{n-d} \to \mathbb{R}^n$ such that there exists a mapping $\kappa : N^{n-2d} \to K(\mathbb{Z}/2,1)$ with the following boundary condition: the restriction $\kappa|_{\partial N} : \partial N \to K(\mathbb{Z}/2,1)$ coincides with the composition $\partial N \subset \mathbb{RP}^{n-d} \subset K(\mathbb{Z}/2,1)$.

Theorem 2. Assuming $n = 4k + (2^\sigma - 1)$, $n = 2^\ell - 1$, $\ell \geq 7$, $\sigma = \left[\frac{\ell-1}{2}\right]$, there exists a mapping $c : S^{n-2k}/\mathbf{i} \to \mathbb{R}^n$ such that the polyhedron L^{n-4k} consists of two components $L^{n-4k}_{\mathbf{Q}}$ and $L^{n-4k}_{\mathbf{H}_b}$. The polyhedron $L^{n-4k}_{\mathbf{Q}}$ is a closed manifold with no boundary. The polyhedron $L^{n-4k}_{\mathbf{H}_b}$ contains boundary $\partial L \subset L^{n-4k}_{\mathbf{H}_b}$. Moreover, the following conditions are satisfied:

–1. There exists a mapping $\zeta_{\mathbf{Q}} : L^{n-4k}_{\mathbf{Q}} \to K(\mathbf{Q},1)$, where $\mathbf{Q}$ is the group of the order 8 of the unite quaternions, $\mathbf{I}_a \subset \mathbf{Q}$ is the subgroup of the order 4 of complex integers, such that the 2-sheeted covering $\bar{L}^{n-4k}_{\mathbf{Q}} \to L^{n-4k}_{\mathbf{Q}}$, which is induced by the 2-sheeted covering $K(\mathbf{I}_a,1) \to K(\mathbf{Q},1)$ over the target space, coincides with the canonical 2-sheeted covering over $L^{n-4k}_{\mathbf{Q}}$.

–2. There exists a mapping $\zeta_{\mathbf{H}_b} : L^{n-4k}_{\mathbf{H}_b} \to K(\mathbf{H}_b,1)$, where $\mathbf{H}_b$ is the group of the order 8, isomorphic to the direct product of the cyclic group $\mathbf{I}_a$ of the orders 4 and the elementary group $\mathbb{Z}/2$ of the order 2. The subgroup $\mathbf{H}_b$ contains the subgroup $\mathbf{I}_a \subset \mathbf{H}_b$. The 2-sheeted covering $\bar{L}^{n-4k}_{\mathbf{H}_b} \to L^{n-4k}_{\mathbf{H}_b}$ with the ramification over ∂L, induced by the 2-sheeted covering $K(\mathbf{I}_a,1) \to K(\mathbf{H}_b,1)$ over the target space coincides with the canonical 2-sheeted covering with the ramification over $L^{n-4k}_{\mathbf{H}_b}$.

The mapping κ constructed in the Theorem 1 is called a relative abelian structure for the mapping d. The pair of mappings $(\zeta_{\mathbf{Q}}, \zeta_{\mathbf{H}_b})$ constructed in Theorem 2 is called a relative quaternionic structure of the mapping c. The theorems are required, in particular, to obtain a new proof the well-known Adams Theorem on Hopf Invariants, see the paper by the author arXiv:1005.1005.

On the KO-theory of toric spaces.

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Central in toric geometry and topology are several important spaces which include moment-angle complexes, the Davis-Januszkiewicz space and toric manifolds. In any complex-oriented cohomology theory, the cohomology rings of many of these spaces have elegant descriptions in terms of the underlying combinatorics. For KO-theory however the situation is more complex. Even so, a surprising amount of the structure does survive from the complex-oriented case. A report of recent joint work with:Luis Astey, Martin Bendersky, Fred Cohen, Don Davis, Sam Gitler, Mark Mahowald, Nigel Ray and Reg Wood.

On new family of explicit Riemannian $SU(4)$-holonomy metrics[1].

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The Calabi metrics founded explicitly in [1] were the first examples of complete Riemannian metrics with $SU(2n)$- and $Sp(n)$-holonomy. These metrics are defined on spaces of $\mathbb{C}$-bundles over the Kähler-Einstein manifold F. We constructs in explicit algebraic form one-parameter family of complete special Kähler metrics, "joinning" these two Calabi metrics in dimension eight for one special choice of F.

Theorem [2]. *For $0 \leqslant \alpha < 1$ every Riemannian metrics of the family*

$$\bar{g}_\alpha = \tfrac{r^4(r^2-\alpha^2)(r^2+\alpha^2)}{r^8-2\alpha^4(r^4-1)-1}dr^2 + \tfrac{r^8-2\alpha^4(r^4-1)-1}{r^2(r^2-\alpha^2)(r^2+\alpha^2)}\eta_1^2 + r^2(\eta_2^2+\eta_3^2)$$

$$+(r^2+\alpha^2)(\eta_4^2+\eta_5^2) + (r^2-\alpha^2)(\eta_6^2+\eta_7^2),$$

is complete smooth metric with $SU(4)$-holonomy on the space of canonical complex line bundle over the manifold of complex 3-flags in $\mathbb{C}^3$. Metric $\bar{g}_0$ is isometric to the Calabi metric [1] with $SU(4)$-holonomy; metric $\bar{g}_1$ is isometric to the Calabi metric [1] on $T^*\mathbb{C}P^2$ with $Sp(2)$-holonomy.

In the above formulas r is radial coordinate in the fibres of bundle, one-form η_1 is dual to angle coordinate on the fibre and one-forms $\eta_2, \ldots, \eta_7$ generates leftinvariant co-frame on F.

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Toric Degenerations and Exact Bohr-Sommerfeld Correspondence

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We consider compact, kähler Hamiltonian toric manifolds, where the underlying integrable systems are smooth, but the associated torus action is singular. Interesting examples are provided by the (classical) Gelfand-Cetlin systems. On such a manifold, there are two natural quantizations possible, one by holomorphic quantization considering sections of a quantizing line bundle, and the other is by the real polarization given by the simultaneous levels of the Hamiltonians at integer values, so-called Bohr-Sommerfeld quantization. The Bohr-Sommerfeld correspondence should show an isomorphism to as high a degree as possible between these two quantizations. The exact Bohr-Sommerfeld correspondence should be a linear map between the two Hilbert spaces giving such an isomorphism. This question was considered a few years ago by Andrey Tjurin.

The Bohr-Sommerfeld condition yields distributional sections to the quantizing line bundle supported on the Bohr-Sommerfeld levels. The natural guess for the implementation for the Bohr-Sommerfeld correspondence would be the Bergman projector from distributional sections to holomorphic suggestions. For toric varieties with smooth Hamiltonians, this is true and easy to see, by character decompositions. For systems like the Gelfand-Cetlin systems this is impossible because the singular torus action is not holomorphic and does not give a representation on the holomorphic sections of the quantizing line bundle. We describe a method to show this which uses degeneration to singular toric varieties, singular algebraic varieties with holomorphic torus action, and a continuity under deformation of integrals of holomorphic sections taken along Bohr-Sommerfeld levels. In passing we discuss geometric quantization for the real polarization given by the torus action, and the relation to classical Delzant theory generalized to singular integrable systems like Gelfand-Cetlin. Relations with classical geodesic flows on rank one symmetric spaces are also discussed.

Parts of this work are joint projects with V. Guillemin and A. Uribe-Ahumada.

Causality in space-times, Low conjecture and the partial order on Legendrian spheres[1]

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Let (X^{m+1}, g) be a space-time, i.e. a time oriented Lorentz manifold. For $x, y \in X$ we say that y is in the causal future of x if there is a future directed nonspacelike curve from x to y. This is denoted by $x \leqslant y$. If the space-time is causal, then the relation $\leqslant$ gives a partial order on it, and we say that x, y are *causally related* if $x \leqslant y$ or $y \leqslant x$.

A space-time X is *globally hyperbolic* if it is causal and the intersection of causal past and causal future of any two points in X is compact. The classical theorem of Geroch combined with the recent results of Bernal and Sanchez show that a globally hyperbolic (X^{m+1}, g) is diffeomorphic to $M \times \mathbb{R}$, where each $M \times t$ is the so called spacelike Cauchy surface of X. The space $\mathcal{N}$ of non-parametrized future directed light rays (null geodesics) in X can be identified with the spherical cotangent bundle ST^*M. All the light rays through $x \in X$ form the Legendrian sphere in $\mathcal{N}$ that is called the *sky* S_x of x.

We consider the space $\mathcal{L}$ of all Legendrian spheres in ST^*M that are isotopic to a fiber of ST^*M and given two such spheres S_1, S_2 we say that $S_1 \leqslant S_2$ if there is a non-negative Legendrian isotopy of S_1 to S_2. We show that if the universal cover of M is not compact, then $S_1 \leqslant S_2$ is indeed a partial order on $\mathcal{L}$ and the inclusion $X \to \mathcal{L}$, $x \to S_x$ preserves the partial order.

This implies that two events x, y in such (X^{m+1}, g) are causally related if and only if the Legendrian link (S_x, S_y) is nontrivial. In the cases where M is an open 2-manifold this gives the proof of the Low conjecture, and when $M = \mathbb{R}^3$ this gives the proof of the Legendrian Low conjecture formulated by Natario and Tod.

Very often the fact that (S_x, S_y) is (topologically) nontrivial can be detected by the generalized linking numbers constructed in our works with Yuli Rudyak.

Legendrian linking is not equivalent to causality when the space-time has a refocussing Lorentz metric. The existence of such a metric seems to be closely related to the Y_l^x Riemann manifold, i.e. manifolds for which there is a point x and a number $l > 0$, such that all the unit speed geodesics starting from x return back to x at time l. If time permits, we will discuss some recent progress relating causality, refocusing and generalizations of Y_l^x-manifolds obtained in the works of Kinlaw, Low, Nemirovski, Rudyak, Sadykov and myself in more detail.

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Real Bott manifolds and acyclic digraphs.

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A *real Bott manifold* is a closed smooth manifold obtained as the total space of an iterated $\mathbb{R}P^1$-bundles starting with a point, where each $\mathbb{R}P^1$-bundle is the projectivization of the Whitney sum of two real line bundles. The diffeomorphism types of real Bott manifolds can be completely characterized real Bott manifolds in terms of three simple matrix operations on square binary matrices symmetrically permutable to strict upper triangular form.

This characterization can be visualized combinatorially in terms of graph operations on directed acyclic graphs. Using this combinatorial interpretation, we prove that the decomposition of a real Bott manifold into a product of indecomposable real Bott manifolds is unique up to permutations of the indecomposable factors.

This talk is based on a part of joint work with Professors M. Masuda and S.-i. Oum.

Enumerative problems for logarithmic forms on hyperplane complements.

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The study of logarithmic vector fields and logarithmic forms on unions of hyperplanes in projective space has a 30-year history that has revealed some interesting subtleties. For example, a well-known formula of Solomon and Terao expresses the characteristic polynomial of the arrangement (matroid) in terms of a specialization of the Hilbert series of modules of logarithmic differentials: however, the Hilbert series of such a module is not uniquely determined by the matroid. Along the same lines, a result of Mustata and Schenck gives the Chern classes of the sheaf of logarithmic 1-forms in terms of the same characteristic polynomial, in the interesting special case where this sheaf is locally free.

I will describe some work that gives new relations amongst the Chern classes of sheaves of logarithmic forms. We see that, in general, they are not uniquely determined by the matroid; however, in some cases one does obtain explicit formulas and a "geometric" explanation of Solomon and Terao's formula in terms of some elementary intersection theory. This is joint work with Mathias Schulze.

Loop spaces for manifolds with group actions[1].

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We construct combinatorial models for loops on manifolds with group actions in terms of piecewise geodesics. In particular case of toric manifolds, these models can be simplified and in certain special cases lead to loop space homology computations.

[1] This is a joint work with Nigel Ray.

Buchstaber Invariant of Simple Polytopes

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Convex polytopes lie in the focus of a scientific study since antiquity. Let us remind the Platonic solids, the Euler-Descartes formula, the Cauchy's and Alexandrov's theorems about unfoldings, the Minkowski's theorem, the Brunn-Minkowski inequality, the g-theorem, and so on. Toric topology gives a new point of view on simple polytopes.

Let $P^n = \{x \in \mathbb{R}^n \colon A_p\boldsymbol{x} + b_p \geqslant 0\}$ be a simple n-polytope and $\mathfrak{F} = \{F_1, \dots, F_m\}$ the set of its facets. Then there is a canonical combinatorial construction of a *moment-angle manifold* $\mathcal{Z}_P$ [BPeng] with a canonical action of a torus T^m such that P^n is an orbit space. Namely, for each facet $F_i \in \mathfrak{F}$ denote by T^{F_i} the one-dimensional coordinate subgroup of $T^{\mathfrak{F}} = T^m$. Then assign to every face G the coordinate subtorus $T^G = \prod_{F_i \supset G} T^{F_i} \subset T^{\mathfrak{F}}$. For every point $q \in P$ let $G(q)$ be a unique face containing q in the relative interior. Then

$$\mathcal{Z}_P = (T^{\mathfrak{F}} \times P^n)/\sim,$$

where $(t_1, p) \sim (t_2, q)$ if and only if $p = q$ and $t_1 t_2^{-1} \in T^{G(p)}$. We have $\mathcal{Z}_P/T^m = P$, and the stabilizer of a point $[(t, q)]$ is $T^{G(q)}$. It turns out that the geometric realization of P gives the realization of $\mathcal{Z}_P$ as a smooth submanifold in $\mathbb{C}^m$ with a trivial normal bundle ([1], see also [BPrus]).

Sometimes there is an $(m-n)$-dimensional subtorus in T^m that acts freely. In this case the orbit space is a smooth $2n$-manifold M^{2n} with a standard action of an n-dimensional torus. Such manifolds are called *quasitoric*. But this is not the general case: there are polytopes that have no quasitoric manifolds at all.

Definition 1. A *Buchstaber number* $s(P)$ is the maximal dimension of a torus subgroup $H \cong T^s$, which acts freely.

It follows from the definition that $s(P)$ is a combinatorial invariant of simple polytopes. In some sense, $s(P)$ is a measure of a symmetry of a moment-angle manifold. In fact, it can be defined for any simplicial complex in such a way that $s(P) = s(\partial P^*)$.

The problem stated by Victor M. Buchstaber in 2002 is to find a simple combinatorial description of the s-number.

At present moment the following problems in this field are actual: to find a simple (or several equivalent simple) combinatorial description that gives an EFFECTIVE method to calculate the s-number in important SPECIAL cases; to find a connection between values of $s(K)$ of different simple polytopes and complexes; to find a connection with other combinatorial invariants.

We study the properties of $s(P)$. It is not difficult to see, that $1 \leqslant s(P) \leqslant m - n$.

Theorem 5. *The s-number satisfies the following properties.*

1. *$s(P)$=1 if and only if P is a simplex;*
2. *For any $k \geqslant 2$ there exists a simple polytope with $m - n = k$ and $s(P) = 2$;*
3. *$s(P) \geqslant m - \gamma + s(\Delta^{\gamma-1}_{n-1})$, where γ is a chromatic number of P, and $\Delta^{\gamma-1}_{n-1}$ is an $(n-1)$-skeleton of a $(\gamma-1)$-dimensional simplex;*

4. *If P is obtained from Q by an i-flip with $2 \leqslant i \leqslant n-1$, then $|s(P)-s(Q)| \leqslant 1$;*

5. *$s(P) \geqslant [\frac{m-n}{2}]$ for a flag polytope;*

6. *There are two polytopes with equal f-vectors and chromatic numbers, but different s-numbers;*

7. *It is known (see [Gb]) that each simple polytope P^n with $m=n+3$ facets can be represented in terms of a regular $(2k-1)$-gon M_{2k-1} and a surjective map from $\mathfrak{F}=\{F_1,\dots,F_{n+3}\}$ to the set of vertices of M_{2k-1}. The facets $F_{i_1},\dots,F_{i_n}$ intersect is a vertex if and only if the triangle formed by the vertices corresponding to the rest three facets contain the center of M_{2k-1}.*

   Let $a_i \geqslant 1$ be the number of the preimages of the i-th vertex of M_{2k-1}. Then for such a polytope $P_{a_1,\dots,a_{2k-1}}$ we have: $s(P)=3$ if and only if $k \leqslant 4$.

   Here k can be expressed in terms of bigraded Betti numbers

$$2k-1=\sum_j \beta^{-1,2j}(\mathcal{Z}_{P_{a_1,\dots,a_{2k-1}}})=\sum_j \beta^{-2,2j}(\mathcal{Z}_{P_{a_1,\dots,a_{2k-1}}}).$$

We also study the properties of simple n-polytopes with $n+3$ facets. Let us denote $\varphi_i = a_i+\dots+a_{i+k-2}$, $\psi_j = a_j+\dots+a_{j+k-1}$, where indices are taken modulo $2k-1$.

Theorem 6. *For the polytope $P = P_{a_1,\dots,a_{2k-1}}$ the bigraded cohomology ring $\mathbb{C}^{*,*}(\mathcal{Z}_P)$ is isomorphic to the free abelian group $\mathbb{Z}\oplus\mathbb{Z}^{2k-1}\oplus\mathbb{Z}^{2k-1}\oplus\mathbb{Z}$ with the generators*

$$\begin{gathered}1,\ \operatorname{bideg} 1=(0,0);\\ X_i,\ \operatorname{bideg} X_i=(-1,2\varphi_i), i=1,\dots,2k-1;\\ Y_j,\ \operatorname{bideg} Y_j=(-2,2\psi_j), j=1,\dots,2k-1;\\ Z,\ \operatorname{bideg} Z=(-3,2(n+3)).\end{gathered}$$

For $k \geqslant 3$

$$X_i\cdot X_j=0 \qquad X_i\cdot Y_j=\delta_{i+k-1,j}Z \qquad Y_i\cdot Y_j=0,$$

and for $k=2$

$$X_i^2=0, \qquad X_iX_{i+1}=-X_{i+1}X_i=Y_i, \qquad X_1X_2X_3=Z.$$

In fact, it is easy to see that $\mathcal{Z}_{P_{a_1,a_2,a_3}} = S^{2a_1-1}\times S^{2a_2-1}\times S^{2a_3-1}$, and according to the results by Lopez de Medrano [LM] for $k \geqslant 3$ the manifold $\mathcal{Z}_{P_{a_1,\dots,a_{2k-1}}}$ is homeomorphic to

$$\mathop{\mathrm{sh}}_{i=1}^{2k-1} S^{2\varphi_i-1}\times S^{2\psi_{i+k-1}-2}.$$

See also [BM]. Our result describes additionally the bigraded structure in the cohomology ring of the moment-angle manifold $\mathcal{Z}_{P_{a_1,\dots,a_{2k-1}}}$.

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Cup Products in Generalized Moment Angle Complexes.

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This is a report of joint work with A. Bahri, M. Bendersky, and F.R. Cohen

The cohomology of a generalized moment angle splits geometrically in terms of smash angle complexes. We define a product on the direct sum of the cohomology groups of these smash moent angle complexes so it becomes a ring which is isomorphic to that of the cohomology ring of the given moment angle complex.

On cohomology length of branched coverings.

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We will talk about so called Dold-Smith branched coverings of topological spaces and the correlation between cohomology (rational and mod p) of the base and the total space of a branched covering. The branched coverings that we consider have been defined by Larry Smith[1] in 1983 as a natural generalization of unbranched coverings on which extends the classical notion of a (co)homological transfer. In 1986 Dold[2] gave the full classification of these branched coverings in terms of actions of finite groups on topological spaces. Subsequently these coverings were called Dold-Smith branched coverings.

Let us give the definitions. All topological spaces are assumed to be Hausdorff. By $\mathrm{Sym}^n X = X^n/S_n$ we denote the n-th symmetric power of a space X. The points of $\mathrm{Sym}^n X$ are precisely the n-multisets $[k_1x_1, \ldots, k_sx_s],\ k_i \in \mathbb{N}, x_i \in X, 1 \leq i \leq s, k_1 + \ldots + k_s = n, x_i \neq x_j, i \neq j$. Let $\exp_n X = \{A \subset X | 1 \leq \sharp A \leq n\}$ be the n-th exponent of X, the space of all finite nonempty subsets of X of order not greater than n. There is a natural "forgetting multiplicities" map $\langle\cdot\rangle : \mathrm{Sym}^n X \to \exp_n X,\ \langle [k_1x_1, \ldots, k_sx_s]\rangle = \{x_1, \ldots, x_s\}$.

Definition 1. Let X and Y be Hausdorff spaces. A continuous map $f : X \to Y$ is called an n-fold Dold-Smith branched covering if there exists a continuous map $g : Y \to \mathrm{Sym}^n X$ such that $f^{-1}(y) = \langle g(y)\rangle\ \forall y \in Y$.

There exists at least three important for topology classes of maps which are Dold-Smith branched coverings:

(i) unbranched n-fold coverings $f : X \to Y$;

(ii) projections $\pi : X \to X/G$ on the orbit spaces of finite group actions, $n = |G|$;

(iii) finite-fold classical branched coverings of smooth manifolds $f : M^m \to N^m,\ n = \max_{y\in N^m}\{\sharp f^{-1}(y)\}$.

Dold's classification theorem states that for every n-fold branched covering $f : X \to Y, g : Y \to \mathrm{Sym}^n X$, there exists a canonically obtained Hausdorff space W with an action of the symmetric group S_n such that $X = W/S_{n-1}, Y = W/S_n$ and $f = \pi_{S_n,S_{n-1}} : W/S_{n-1} \to W/S_n$ is a natural orbit projection.

Using Dold's result and classical cohomology transfer for finite group actions one can observe that for any n-fold Dold-Smith branched covering $f : X \to Y, g : Y \to \mathrm{Sym}^n X$ of "good" spaces (it's sufficient X, Y to be both locally contractible metric spaces or both countable CW-spaces) the induced homomorphisms $f^* : H^*(Y;\mathbb{Q}) \to H^*(X;\mathbb{Q})$ and $f^* : H^*(Y;\mathbb{Z}_p) \to H^*(X;\mathbb{Z}_p),\ p > n$, in singular cohomology are *monomorphisms*. So cohomology of Y is always a subalgebra of cohomology of X. The question that we answer in this talk is how small ("degenerate") can be the subalgebra $H^*(Y;K) \subset H^*(X;K),\ K = \mathbb{Q}$ or $\mathbb{Z}_p,\ p > n$, when we fix $H^*(X;K)$ and the number of sheets n. It turns out that the proper notion of "richness" or "smallness" of an graded algebra is its multiplicative length (= the cohomology length of the underlying space). Denote by $l(X)$ the rational cohomology length of a space X, $l_p(X)$ — mod p cohomology length of X.

Theorem. *Let $f : X \to Y, g : Y \to \mathrm{Sym}^n X$ be an n-fold branched covering of locally contractible paracompact spaces such that $Y \times X^n$ is also paracompact. Then the following estimate holds: $l(Y) + 1 \geq \frac{l(X)+1}{n},\ l_p(Y) + 1 \geq \frac{l_p(X)+1}{n}, \forall p > n$. These estimates are sharp for $n = 2$.*

The proof of the theorem has required a new algebraic notion of so called graded

Frobenius n-homomorphisms. The theory of (ungraded) Frobenius n-homomorphisms was built by V.M.Buchstaber and E.G.Rees starting from 1996. The graded Frobenius n-homomorphisms are special linear maps $f : A^* \to B^*$ of graded associative commutative algebras over a (graded) ground ring R^* for which holds a special "weak multiplicativity" axiom. 1-homomorphisms are just algebra homomorphisms. The "weak multiplicativity" axiom for 2-homomorphisms can be written in the way $f(abc) = -\frac{1}{2}f(a)f(b)f(c) + \frac{1}{2}(f(a)f(bc) + f(b)f(ca) + f(c)f(ab))$ (it is the case when $a, b, c \in A^*$ are of even degree, in the cases of other degrees one needs to put another signs in the right side of the formula). It can be proved that the sum $f = f_1 + \ldots + f_n : A^* \to B^*$, where $f_i : A^* \to B^*, 1 \leq i \leq n$, are algebra homomorphisms, is an n-homomorphism. So the sum of n algebra homomorphisms inherits some "weak multiplicativity". This fact applied to the transfer in cohomology of Dold-Smith branched coverings with additional algebraic technique was used to prove the above theorem.

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Equivariant Schubert calculus of Coxeter group $I_2(m)$.

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Let G be a Lie group and T be its maximal torus. The homogeneous spaces G/T is known to be a smooth variety and called the *flag variety* of type G. Its cohomology group has a distinguished basis consisting of *Schubert classes*, which arise from a certain family of sub-varieties. The ring structure of $H^*(G/T)$ with respect to this basis reveals interesting interactions between topology, algebraic geometry, representation theory, and combinatorics, and has been studied under the name of *Schubert calculus*.

One way to study $H^*(G/T)$ is to identify it with the coinvariant ring of the Weyl group W of G, i.e. the polynomial ring divided by the ideal generated by the invariant polynomials of W. From this point of view, the problem can be rephrased purely in terms of W and extended to any Coxeter group including non-crystallographic ones. In fact, H. Hiller pursued this way in his book "The geometry of Coxeter groups" and gave a characterization of a "Schubert class" in the coinvariant ring.

On the other hand, G/T has the canonical action of T and we can consider the equivariant topology with respect to this action. A similar story goes for the equivariant cohomology $H_T^*(G/T)$ and we can consider *equivariant Schubert calculus* for Coxeter groups. This time we consider a double version of a coinvariant ring. Along this line, the first difficulty is how to find polynomials in it representing Schubert classes. By several people, such polynomial representatives have been found for type A_n, B_n, C_n, D_n. Here we give polynomial representatives for the non-crystallographic group of type $I_2(m)$. The main ingredients is the localization technique, a powerful machinery of equivariant topology.

Estimates of Z_2-index of the grassmanian G_{2n}^{n}[1].

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The topology of the real Grassmannians has many applications in the discrete and convex geometry. For example, some topological facts were applied to obtain some existence theorems for flat transversals (affine flats intersecting all members of a given family of sets) in the works of R. Živaljević, S.T. Vrećica, V.L. Dol'nikov.

In this talk we consider the Grassmannian G_{2n}^{n} of n-dimensional subspaces of $\mathbb{R}^{2n}$. This space has a natural Z_2-action (involution) by taking the orthogonal complement of the subspace. The well-known invariant of Z_2-spaces is homological index, introduced and studied by Krasnosel'skii, Schwarz, Conner and Floyd. This invariant proved to be very useful in applications to combinatorics and convex geometry.

The following theorem gives an estimate for the index of the Grassmannian.

Theorem. *If $n = 2^l(2m+1)$, then*

$$2^{l+1} - 1 \leq \operatorname{ind} G_{2n}^{n} \leq 2n - 1,$$

for $n = 2m+1$ the index equals 1, for $n = 2(2m+1)$ the index equals 3.

The lower and the upper bounds coincide for $n = 2^l$, odd n, $n = 2(2m+1)$. In other cases there is still some gap between them. This result easily produces some geometric consequences. Here is one example.

Corollary. *Let $n = 2^l(2m+1)$, $k = 2^{l+1} - 1$. Consider some k continuous (in the Hausdorff metric) $O(n)$-invariant functions $\alpha_1, \dots, \alpha_k$ on (convex) compacts in $\mathbb{R}^n$. Then for any (convex) compact $K \subseteq \mathbb{R}^{2n}$ there exist a pair of orthogonal n-dimensional subspaces L and M, such that for their respective orthogonal projections π_L and π_M we have*

$$\forall i = 1, \dots, k \ \alpha_i(\pi_L(K)) = \alpha_i(\pi_M(K)).$$

In this corollary α_i can be the Steiner measures (volume, the boundary measure, etc.), for example.

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Moment Polyhedra, Semigroup of Representations and Kazarnovskii's Theorem.

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Two representations of a reductive group G are *spectrally equivalent* if the same irreducible representations appear in both of them. The semigroup S of finite-dimensional representations of G with tensor product and up to spectral equivalence is a rather complicated object. The Grothendieck group of S contains significant information about S and is simpler to describe. In the talk I will give its description in terms of *moment polyhedra* of representations.

As a corollary, one can obtain the Kazarnovskii theorem ([1]) on the number of solutions in G of a system $f_1 = \cdots = f_m = 0$ where $m = \dim(G)$ and each f_i is a generic function in the space of matrix elements of a representation π_i of G. Given a representation π of a classical group G one defines its *Newton polyhedron* $\tilde{\Delta}(\pi)$ fibred over the moment polyhedron $\Delta(\pi)$ with *Gelfand-Cetlin polyhedra* as fibres. Then, for classical groups, the Kazarnovskii theorem can be formulated exactly as the famous Bernstein-Kushnirenko theorem from the Newton polyhedra theory: *the number of solutions of the system under discussion is equal to the mixed volume of the Newton polyhedra* $\tilde{\Delta}(\pi_i)$ *multiplied by* $m!$. The proof is based on the intersection theory for finite-dimensional subspaces of rational functions on algebraic varieties (see [2]-[4]).

My talk is based on a joint work with Kiumars Kaveh ([5]).

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On cohomological rigidities of toric hyperKähler manifolds.

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In [6], Masuda proved the following theorem:

Theorem 1 (Masuda)**.** Two toric manifolds (M, T) and (M', T) are weakly equivariantly isomorphic as varieties if and only if $H_T^*(M;\mathbb{Z})$ and $H_T^*(M';\mathbb{Z})$ are weakly $H^*(BT)$-algebra isomorphic.

Motivated by this Masuda's theorem, Masuda and Suh proposed the following problem in [8]:

Problem 1 (Cohomological rigidity problem)**.** Let M and M' be (quasi)toric manifolds. Are they homeomorphic (or diffeomorphic) if $H^*(M) \simeq H^*(M')$?

This problem is still open, but this can be asked for the class of other manifolds. For example, this problem can be asked for more general *torus manifolds* or *small covers* which are the real analogue of quasitoric manifolds; however, the answers for both classes are negative (see [3, 7]). In this talk, we study the cohomological rigidity problem for *toric hyperKähler manifolds.*

Toric hyperKähler manifolds, introduced by Bielawski and Dancer in [2], are defined by the hyperKähler quotient of torus actions on quaternionic spaces. This manifold can be regarded as the hyperKähler analogue of the symplectic toric manifolds.

In this talk, we show the following theorem.

Theorem 2 ([3])**.** Let $(M_\alpha, T, \mu_{\widehat{\alpha}})$ and $(M'_{\alpha'}, T, \mu'_{\widehat{\alpha}'})$ be triples of toric hyperKähler manifolds with torus actions and their hyperKähler moment maps. Then, $(M_\alpha, T, \mu_{\widehat{\alpha}})$ and $(M'_{\alpha'}, T, \mu'_{\widehat{\alpha}'})$ are weakly hyperhamiltonian isomorphic if and only if there is a weak $H^*(BT)$-algebra isomorphism $f : H_T^*(M_\alpha;\mathbb{Z}) \to H_T^*(M'_{\alpha'};\mathbb{Z})$ such that $f(\widehat{\alpha}) = \widehat{\alpha}'$.

Here, we call two triples $(M_\alpha, T, \mu_{\widehat{\alpha}})$ and $(M'_{\alpha'}, T, \mu'_{\widehat{\alpha}'})$ are *weakly hyperhamiltonian isomorphic* if there is a diffeomorphism $f : M \to M'$ such that

- f is weakly equivariant map, i.e., there is an isomorphism $\varphi : T \to T$ such that $f(xt) = f(x)\varphi(t)$, where $x \in M$ and $t \in T$;
- f preserves (weak) hyperhamiltonian structures, i.e., f preserves hyperKähler structures on M and M' and the following diagram is commute:

$$\begin{array}{ccc} M & \stackrel{\mu_{\widehat{\alpha}}}{\to} & \mathfrak{t}^* \oplus \mathfrak{t}^*_{\mathbb{C}} \\ f\downarrow & & \downarrow \varphi^* \\ M' & \stackrel{\mu'_{\widehat{\alpha}'}}{\to} & \mathfrak{t}^* \oplus \mathfrak{t}^*_{\mathbb{C}}, \end{array}$$

  where $\mathfrak{t}^*$ is the dual of Lie algebra of T, $\mathfrak{t}^*_{\mathbb{C}}$ is its complexification, and $\varphi^* : \mathfrak{t}^* \oplus \mathfrak{t}^*_{\mathbb{C}} \to \mathfrak{t}^* \oplus \mathfrak{t}^*_{\mathbb{C}}$ is the induced isomorphism from φ.

Moreover, we show the following theorem.

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Theorem 3 ([5]). Let M and M' be toric hyperKähler manifolds. Then, M and M' are diffeomorphic if and only if $\dim M = \dim M'$ and $H^*(M) \simeq H^*(M')$.

Theorem 2 can be regarded as the hyperKähler version of the Masuda's theorem, and Theorem 3 gives the answer of the cohomological rigidity problem for toric hyperKähler manifolds. That is, Theorem 3 says that the cohomological rigidity does not hold for the set of all toric hyperKähler manifolds, but holds for the set of $4n$-dimensional toric hyperKähler manifolds (if n is fixed).

Due to the Bielawski's results in [1], as a corollary of Theorem 3, we have the following result:

Corollary 1. *Let $\mathfrak{M}_n$ be the set of $4n$-dimensional, simply connected, complete, hyperKähler manifolds with effective n-dimensional hyperhamiltonian torus actions. Then $\mathfrak{M}_n$ satisfies cohomological rigidity.*

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Almost complex quasitoric manifolds

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We show that existence and properties of invariant almost complex structures on quasitoric manifolds are ruled by combinatorial invariants corresponding to these manifolds. This allows to obtain an upper bound for number of almost complex structures on a quasitoric manifold M^{2n}: it can't exceed 2^n.

Finite group actions on aspherical spaces.

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A new method of studying finite group actions on aspherical spaces is proposed. Some interesting results will be shown:

1. Complete homotopy classification of finite group free actions on aspherical spaces:

   Theorem: Let G be a finite group. Then the set $\mathbb{S}$ of all free actions of G (up to homotopy conjugation) on an Eilenberg-MacLane space $K(\pi, 1)$ is in one-to-one correspondence with a set of all extensions $1 \to \pi \to S \to G \to 1$ of G by π [1]. The classification of such extensions is determined by group cohomologies $H^*(G)$ (see for example [1]).

   There are also some results on the classification of non-free actions on $K(\pi, 1)$.

2. A connection between a structure of a subgroup lattice of a finite group G and group cohomologies $H^*(G)$ is found. The connection is expressed in terms of Hochschild-Serre spectral sequences. This result is obtained by using theory of classifying spaces of small categories that was introduced by Segal and Quillen (see [2, 3]).

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[1] Some natural conditions (like existance of CW-complex structure on $K(\pi, 1)$) are used.

Intersection of quadrics, moment-angle manifolds and connected sums.

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Intersections of quadrics in R^m given by equations of the form

$$Z = \{\sum_{i=1}^{n} \Lambda_i x_i^2 = 0, \quad \sum_{i=1}^{n} x_i^2 = 1\}$$

including their important complex versions (which are essentially the same as those called now *moment-angle manifolds*):

$$Z^{\mathbb{C}} = \{\sum_{i=1}^{n} \Lambda_i |z_i|^2 = 0, \quad \sum_{i=1}^{n} |z_i|^2 = 1\}$$

(where the coefficients $\Lambda_i \in R^k, i = 1, \ldots, m$ satisfy a generic property) have been studied from the point of view of Geometric Topology since the 80's when they appeared (and keep reappearing since then) in problems of Singularities of Mappings, Dynamical Systems and Algebraic Geometry and are related to many other geometric theories.

The topology of Z for the case $k = 2$ was studied in [LdM1], [LdM2] where it was shown that they are in most cases diffeomorphic to a triple product of spheres or to the connected sum of sphere products. The proof relied heavily on a normal form for them and involved many computations. A geometric description of the group actions on them and of their polytope quotients as well as that of the homology of those manifolds was equally valid for the intersection of any number of such quadrics, but the obstacle to extending the main result for more than two turned out to be the hopeless-looking problem of finding their normal forms, close to that of classifying all simple polytopes.

Their study continued in other directions, especially to the projectivizations (known now as *LV-M manifolds*) of the manifolds $Z^{\mathbb{C}}$, which produced many new examples of non-algebraic complex manifolds fibering over toric varieties (see [Me-V] for a review). Following these lines, in [B-M] a deep study of LV-M manifolds included important advances on the topology of the manifolds $Z^{\mathbb{C}}$ for $k > 2$. The main questions addressed in this respect were the following:

1) Whether they can always be built up from spheres by repeatedly taking products or connected sums: they produced new examples for any k which are so, but also showed how to construct many cases which are not. Many interesting questions arose, including a specific conjecture.

2) The question of the transition between different topological types when the generic condition is broken at some point (*wall-crossing*).

3) A product rule of their cohomology ring (in the spirit of the description of the homology of Z given in [LdM2]) and its applications to question 1).

Meanwhile, and independently, in [D-J] essentially the same manifolds were constructed in a more abstract way, where the main objective was to study the algebraic topology of some important quotients of them called initially *toric manifolds* and now *quasitoric manifolds*. This article originated an important development through the work of many authors, and there is a vast and deep literature along those lines for which the reader is referred to [B-P2]. Yet for a long time no interchange occurred between the two lines of research involving the same objects, until a small connection appeared in the final version of [B-M].

In particular, it turned out that examples relevant to question 1) above were known to these authors (see, for example, [Ba2]), and in [Ba1] there is a product rule for the cohomology ring, similar but dual to that of [B-M] mentioned in 3) above. Those examples were independent and more or less simultaneous to those of [B-M] and meant to answer different questions, but both product rules are consequences of an earlier description of the cohomology ring by Buchstaber and Panov, the first version of which was announced in 1998 ([B-P1]).

One recent expression of this line of research is the article [B-B-C-G] where a far-reaching generalization is made and a general splitting formula is derived. This understanding was fundamental in the process of tracing a way for the case of $k > 2$ quadrics.

The results obtained (in collaboration with Samuel Gitler) follow the three lines described above, but including now all the manifolds Z and not only the moment-angle manifolds:

1) The identification of very general families of them which are diffeomorphic to connected sums of sphere products, including those conjectured in [B-M].

2) The explicit topological description of some of the transitions.

3) The computation of the cohomology ring of an important example that shows that the product rules have to be modified in the general version.

Nevertheless, the final proofs of these results do not depend logically on [LdM2] or [B-B-C-G]. Several new questions and conjectures have arisen.

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A differential operator and tom Dieck-Kosniowiski-Stong localization theorem.

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Abstract. We define a differential operator on the cohomology of the classifying space of 2-torus group, and study the relationship between this operator and tom Dieck-Kosniowiski-Stong localization theorem. As a further application, we determine the group structure of equivariant cobordism classes of all 4-dimensional 2-torus manifolds, and show that each equivariant cobordism class in all 4-dimensional 2-torus manifolds contains a small cover as its representative.

Quasimorphisms, random walks, and knots.

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We study the probabilistic behaviour of quasimorphisms of groups with respect to random walks on groups. Our results have corollaries for braid groups and knots (see Theorem 1).

Definition: completely transient subsets. Let G be a countable (discrete) group, let S be a subset of G, and let μ be a probability measure on G. We say that S is *uniformly transient with respect to* μ if there exists a constant $C := C(\mu)$ such that for any $g, h \in G$ we have

$$\sum_{k=0}^{+\infty} \mu^{*k}(gSh) \; < \; C,$$

where μ^{*k} is the k-fold convolution of μ. (In particular, if S is a uniformly μ-transient set, then $\mu^{*k}(S)$ tends to 0 as k tends to ∞, so S is clearly a "small" set in a certain sense.)

We say that S is *completely transient* if it is uniformly transient with respect to all *nondegenerate* probability measures on G. (A measure on G is said to be *nondegenerate* if its support generates G as a semigroup.)

Theorem 1. In the Artin braid group B_n with $n \geq 3$ the following subsets are completely transient:

1. The set of all non-pseudo-Anosov braids (i. e., the set of all braids of periodic or reducible type in terms of the Nielsen–Thurston classification).
2. The set N^k, where N is the set of all non-pseudo-Anosov braids (for any $k \in \mathbb{N}$).
3. The set of those braids that represent[1] non-hyperbolic knots or links. (In particular, the sets of those braids that represent trivial, non-prime, composite, split, satellite, torus knots or links are completely transient.)
4. The set of those braids that represent knots of genus $\leq k$ (for any $k \in \mathbb{N}$).
5. The set of non-minimal[2] braids.

[1]We consider the classical representation of (oriented) knots and links by braids in the sense of J. W. Alexander, A. A. Markov.

[2]A braid $\beta \in B_n$ is said to be *minimal* if the link represented by β is not represented by braids from B_{n-1}.

Signature of manifolds with proper action of a discrete group and the Hirzebruch type formula.

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V.A.Roklin was the first ([1]) who has written the formula for the signature of 4-dimensional manifolds in the terms of the Pontryagin classes. For manifolds of arbitrary dimension this formula is known as the Hirzebruch formula. The formula was generalized during throughout more than 50 years in various directions.

Here we consider a case of manifolds with proper action of a discrete group G, that is if for any point its isotropy subgroup is finite and the quotient space is compact. It is a natural generalization of the category of non simply connected compact manifolds where a variety of geometric and topological constructions can be extended.

In particular on the category of manifolds with proper action one can canonically construct a bordism relation. For that category in the paper by P.Baum, A.Connes and N.Higson ([2]) a universal space was constructed to which any manifold with proper action of discrete group can be mapped equivariantly up to equivariant homotopy. Due to papers by S.Illman ([3]) and T.Korppi ([4]) we know that any smooth proper action is simplicial with respect to a simplicial structure on the manifold M. It allows to extend for proper actions many combinatorial constructions and to construct correspondent invariants.

Simplicial structure on the manifold with proper action of a discrete group G allows to construct so called algebraic Poincare complex (APC). In particular the APC has noncommutative (symmetric) signature as an element of Hermitian K–theory of the group G, $\mathbf{sign}\,(M) \in \mathbf{K}^*(\mathbf{Q}[\mathbf{G}])$. $\mathbf{sign}\,(M)$ is both homotopy invariant of the manifold M and invariant of bordisms.

Hence the problem of search of the Hirzebruch type formula for the signature $\mathbf{sign}\,(M)$ arises in the terms of the feasible characteristic classes of the quotient space M/G. The trouble is that the quotient space is manifold with singularities. But one can show that the space M/G is the Poincare space for rational homology and the Pontryagin classes has representations as invariant differential forms relative to proper action. It allows to express usual signature of the quotient space M/G by means of the Hirzebruch type formula.

For noncommutative signature $\mathbf{sign}\,(M) \in \mathbf{K}^*(\mathbf{Q}[\mathbf{G}])$ one need to restore a bundle on the quotient space M/G with structural group $GL(n, C^*[G])$, the analog of canonical bundle $\xi_{C^*[G]} \in K_{C^*[G]}(BG)$, that is defined by a natural representation of the group G into the group C^*-algebra $C^*[G]$.

To clarify the bordism concept for proper action one can apply so called the Conner-Floyd construction for fixed points. Calculation of equivariant bordisms for manifolds with proper action is reduced to description of the classifying space for equivariant vector bundles for the case of quasi-free action of the group G on the base ([5]).

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Topological Transversals

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We say that F has a topological ρ-transversal of index (m, k), $\rho < m, 0 < k \leqslant d-m$, if there are, homologically, as many transversal m-planes to F as m-planes through a fixed ρ-plane in $\mathbb{R}^{m+k}$.

Clearly, if F has a ρ-transversal plane, then F has a topological ρ-transversal of index (m, k), for $\rho < m$ and $k \leqslant d - m$. The converse is not true. It is easy to give examples of families with a topological ρ-transversal but without a ρ-transversal plane. We conjecture that for a family F of $k + \rho + 1$ compact, convex sets in euclidean d-space $\mathbb{R}^d$, there is a ρ-transversal plane if and only if there is a topological ρ-transversal of index (m, k). The purpose of this paper is to prove some importante cases of this conjecture and to use them, together with the Lusternik-Schnirelmann category and several versions of the colorful Helly Theorem of Lovasz, to obtain geometric results that, until know, can not be obtained by us only with geometric tools.

A system Ω of λ-planes in $\mathbb{R}^d$ is a continuous selection of a unique λ-plane in every direction of $\mathbb{R}^d$. More precisely, it is a continuous function $\Omega : G(d, \lambda) \longrightarrow M(d, \lambda)$ with the property that $\Omega(H)$ is parallel to H, for every $H \in G(d, \lambda)$.If $\gamma^{d,\lambda} : E^{d,\lambda} \longrightarrow G(d, \lambda)$ is the standard vector bundle of all λ-planes through the origin in $\mathbb{R}^d$, then a system of λ-planes is just a section $s : G(d, d-\lambda) \longrightarrow E^{d,d-\lambda}$ for the vector bundle $\gamma^{d,d-\lambda}$.

We use the notions of topological transversal and of system Ω of λ-planes in $\mathbb{R}^d$ to obtain geometric transversal results.

On the isometries of foliated manifolds[1].

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Let M, N be n - dimensional smooth manifolds on which there are given $k-$ dimensional smooth foliations F_1, F_2 respectively (where $0 < k < n$).

If for the some C^r- diffeomorphism $f : M \to N$ the image $f(L_\alpha)$ of any leaf L_α of foliation F_1 is a leaf of foliation F_2, we say that pairs (M, F_1) and (N, F_2) C^r- diffeomorph. In this case the mapping f is called C^r- diffeomorphism, preserving foliation and is written as

$$f : (M, F_1) \to (N, F_2).$$

In the case $M = N$, $F_1 = F_2$ f is said diffeomorphism of foliated manifold (M, F).

Diffeomorphisms, preserving foliation, are investigated in [1], [2].

Definition.[3] Diffeomorfism $\varphi : M \to M$ a class $C^r(r \geqslant 0)$, preserving foliation, is called an foliation isometry F (an isometry of foliated manifold (M, F)) if it is an isometry on each leaf foliation F, i.e. for each leaf L_α foliation $\varphi : L_\alpha \to \varphi(L_\alpha)$ is an isometry.

Papers [4], [5] are devoted to isometric mappings of foliations. In these papers it is investigated question under what conditions any isometry of the foliation is an isometry of manifold and it is proved the existence of diffeomorfism of foliated manifold on itself which is an isometry of foliation, but it is not an isometry of manifold. It is constructed the example of diffeomorfism of three - dimensional sphere which is the isometry of Hopf fibration but is not an isometry of three - dimensional sphere.

Let M be a n- dimensional smooth connected Riemannian manifold with Riemannian metric g, F-smooth $k-$ dimensional foliation on M (In this paper manifolds and foliations have smoothness C^∞). We denote through $L(p)$- a leaf of foliation F passing through point p, T_pF- tangent space to the leaf $L(p)$ at p and H_pF- it's orthogonal complement of T_pF in $T_pM, p \in M$. We get two subbundles (smooth distributions) $TF = \{T_pF : p \in M\}$, $\quad HF = \{H_pF : p \in M\}$ of tangent bundle TM of manifold M, and as the result tangent bundle TM of manifold M decomposing in the sum of two orthogonal bundles, i.e. $TM = TF \oplus HF$. Restriction of Riemannian metric g on T_pF for all p induces Riemannian metric on the leaves. Induced Riemannian metric defines distance function on every leaf. Further everywhere in this paper under the distance on a leaf is understood this distance. This distance on a leaf different from distance induced by the distance on M.

Let's denote as $G^r_F(M)-$ the set of all C^r isometries of foliated manifold (M, F), where $r \geqslant 0$. Following remarks show that notion of isometry of foliated manifold is correctly defined.

Remark 1. If $r \geqslant 1$, for each element $\varphi \in G^r_F(M)$ the differential $d\varphi$ preserves the length of each tangent vector $\nu \in T_pF$, i.e. holds $\mid d\varphi_p(\nu) \mid = \mid \nu \mid$ at any $p \in M$.

Remark 2. If $r = 0$ each element φ from $G^r_F(M)$ is homeomorphism of manifold M. Riemannian metric of the manifold M induces Riemannian metric on each leaf L_α which defines distance on it. In this case φ is an isometry between metric spaces L_α and $\varphi(L_\alpha)$. Then according to the known theorem, φ is a diffeomorphism of L_α on $\varphi(L_\alpha)$ for each leaf L_α and it's differential

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preserves the length of each tangent vector $\nu \in T_pF$, i. e. holds $| d\varphi_p(\nu) |=| \nu |$ at any $p \in M$ [5, the page 74]. But as shown in the simple example, from differentiability of mapping on each leaf does not follow it's differentiability on all manifold M .

The set $Diff^r(M)$ of all diffeomorphisms of manifold M onto itself is the group related to composition and inverse mapping. The set $G_F^r(M)$ is a subgroup of group $Diff^r(M)$.

The purpose of our paper is to study the group $G_F^r(M)$ with some topology on set $G_F^r(M)$ which has been introduced in the paper [6], depending on the foliation F, such that it coincides with compact open topology when F is n- dimensional foliation. If codimension of foliation F is equal to n , convergence in our topology coincides with pointwise convergence.

Let $\{K_\lambda\}$ be a family of all compact sets where each K_λ is a subset of some leaf of foliation F and let $\{U_\beta\}-$ family of all open sets on M. We consider for each pair K_λ and U_β set of all mappings $f \in G_F^r(M)$, for which $f(K_\lambda) \subset U_\beta$. This set of mappings we denote through $[K_\lambda, U_\beta] = \{f : M \to M | f(K_\lambda) \subset U_\beta\}$.

It isn't difficult to show that every possible finite intersections of sets of the form $[K_\lambda, U_\beta]$ forms a base for some topology. This topology we call foliated compact open topology or in brief $F-$ compact open topology.

Proposition. The set $G_F^r(M)$ with $F-$ compact open topology is Hausdorff space.

The following theorem shows some property of group $G_F^r(M)-$ with $F-$ compact open topology.

Theorem 1. Let M- complete smooth n dimensional manifold with smooth k dimensional foliation F, $f_m \in G_F^r(M), \quad r \geqslant 0, \quad m = 1, 2, 3, ...$ Suppose, that for each leaf L_α there exists a point $o_\alpha \in L_\alpha$ such that the sequence $f_m(o_\alpha)$ is convergent. Then there exists a subsequence f_{m_l} of the sequence f_m which converges in $F-$ compact open topology.

Theorem 2. Let $M-$ smooth complete Riemannian manifold of dimension n with smooth foliation F of dimension k, where $0 < k < n$. Then

1) Each leaf with induced Riemannian metric is complete Riemannian manifold.

2) Let $\gamma_m : R^1 \to L_m$ - sequence of geodesics (of determined by induced Riemannian metrics) on leaves L_m. If $\gamma_m(s_0) \to p$ at $m \to \infty$ for the some $s_0 \in R^1$, then there exists subsequence γ_{m_l} of sequence γ_m which pointwise converges to some geodesic $\gamma : R^1 \to L(p)$ of leaf $L(p)$, passing through the point p at $s = s_0$.

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Links of graphs.

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The talk will be devoted to a review of the today's states of the problems of the homotopy and isotopy classifications of links of finite graphs in the three-dimensional sphere.

Constructions of 3-dimensional small covers.

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Small covers were introduced by Davis and Januszkiewicz [1] as n-dimensional closed manifolds M^n with a locally standard $(\mathbb{Z}_2)^n$-action such that its orbit space is a simple convex polytope. In this talk we are interested in constructions of 3-dimensional small covers M^3 by using operations called a *connected sum* $\sharp$ and a *surgery* $\natural$.

In [2] Izmestiev studied a special class of 3-dimensional small covers which are called *linear models*. He proved that each linear model can be constructed from the 3-dimensinal torus T^3 by using three operations $\sharp$, $\natural$ and $\natural^{-1}$. In [4] Lü and Yu considered a construction of general 3-dimensional small covers. They introduced new operations $\sharp^e$, $\sharp^{eve}$, $\sharp^{\Delta}$ and $\sharp_i^C$ and showed the following theorem.

Theorem (1) (Lü and Yu)**.** Each small cover M^3 can be constructed from $\mathbb{R}P^3$ and $S^1 \times \mathbb{R}P^2$ by using seven operations $\sharp$, $\natural^{-1}$, $\sharp^e$, $\sharp^{eve}$, $\sharp^{\Delta}$, $\sharp_4^C$ and $\sharp_5^C$.

In [3] Kuroki pointed out that the operations $\sharp^e$ and $\sharp^{eve}$ are obtained as compositions of $\sharp$ and $\natural$. In this talk we improve the above theorem as follows.

Theorem (2)**.** (1) Each small cover M^3 can be constructed from T^3, $\mathbb{R}P^3$ and $S^1 \times \mathbb{R}P^2$ with two different $(\mathbb{Z}_2)^3$-actions by using two operations $\sharp$ and $\natural$.
(2) Each small cover M^3 can be constructed from $\mathbb{R}P^3$ and $S^1 \times \mathbb{R}P^2$ with two different $(\mathbb{Z}_2)^3$-actions by using four operations $\sharp$, $\sharp^e$, $\natural^{-1}$ and $\sharp_4^C$.

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Polynomially Bounded Cohomology and the Strong Novikov Conjecture.

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In 1988, Connes and Moscovici proved the following:

Theorem - [CM] *Suppose a countable discrete group π satisfies*

<u>Condition PC</u> *Every rational cohomology class of π is represented by a cocycle of polynomial growth.*

<u>Condition RD</u> *The group π is RD (rapid decay).*

Then the assembly map for the topological K-groups of the reduced C^*-algebra $C^*_r(\pi)$ is rationally injective.

The condition PC is easily expressed in terms of the standard representation of group cocycles. The second condition is more technical, dealing with the existence of a certain type of subalgebra of $C^*_r(\pi)$. It is also apparently more restrictive, since the known examples of groups satisfying condition RD also satisfy condition PC.

We show that Condition RD above can be removed without affecting the conclusion of the theorem. First, some terminology. Let (π, L) be a discrete group equipped with word-length function L, and let $\mathcal{P}H^*(\pi) = \mathcal{P}H^*(\pi;\mathbb{C})$ denote the polynomially bounded cohomology of (π, L) with coefficients in the trivial π-module $\mathbb{C}$. There is a <u>comparison map</u>

$$\Psi = \Psi^*_{\mathcal{P}} : \mathcal{P}H^*(\pi) \to H^*(\pi) \tag{9}$$

whose image consists of those classes which can be represented by a group cocycle which is polynomially bounded with respect to the word-length function L on π and the standard norm on $\mathbb{C}$. Let

$$\mathcal{A}_\pi : \bigoplus_{m\geq 0} H_{*-2m}(\pi;\mathbb{Q}) \to K^t_*(C^*_r(\pi)) \otimes \mathbb{Q}$$

denote the assembly map going from the (rationalized) connective K-homology groups of $B\pi$ to the (rationalized) topological K-groups of $C^*_r(\pi)$).

Theorem A *For each $H^n(\pi) \ni [c] \in im(\Psi)$, there is a map $\varphi_{[c]} : K^t_*(C^*_r(\pi)) \otimes \mathbb{Q} \to \mathbb{C}$ for which*

$$\varphi_{[c]}(\mathcal{A}_\pi(x_n, x_{n-2}, x_{n-4}, \dots)) =< c, x_n > \in \mathbb{C} \tag{10}$$

Observing that the construction of $\varphi_{[c]}$ arises via the extension of a cyclic group cocycle originally defined over the complex group algebra, the Index Theorem of Connes-Moscovici (following that of Mishchenko-Fomenko) allows us to reformulate the above result as

Theorem A$'$ *Let M be a closed, compact, oriented n-dimensonal manifold, $\mathcal{L}(M)$ its total Hirzebruch L-class, $[M]$ its fundamental homology class, and $\iota : M \to B\pi_1(M)$ the classifying map for the fundamental group of M. Then the higher signatures*

$$Sign_c(M) := \langle \mathcal{L}(M)\iota^*(c), [M]\rangle \in \mathbb{Q}$$

are invariants of the orinted homotopy type of M whenever $[c] \in H^*(\pi;\mathbb{Q})$ is represented by a cocycle whose growth - with respect to a word-length function on π and the standard norm on $\mathbb{Q}$ - is at most polynomial.

It is an easy to show using Bott periodicity that if the assembly map $\mathcal{A}_\pi$ is injective, so is the full (rational) assembly map

$$\widetilde{\mathcal{A}}_\pi : \bigoplus_{m\in\mathbb{Z}} H_{*-2m}(\pi;\mathbb{Q}) \to K_*^t(C_r^*(\pi)) \otimes \mathbb{Q} \tag{11}$$

Consequently,

Corollary B *If every rational homology class of π pairs non-trivially with a cocycle of polynomial growth, then the full assembly map $\widetilde{\mathcal{A}}_\pi$ is injective.*

To describe a class of groups for which Corollary B applies, we recall that π is said to be $\mathcal{P}$-isocohomological (or $\mathcal{P}$-IC) if the comparison map in (9) is an isomorphism, and strongly $\mathcal{P}$-isocohomological ($\mathcal{P}$-SIC) if the comparison map

$$\Psi^*_{\mathcal{P}}(\pi;V) : \mathcal{P}H^*(\pi;V) \to H^*(\pi;V) \tag{12}$$

is an isomorphism for all bornological $H_L^{1,\infty}(\pi)$-modules V, where $H_L^{1,\infty}(\pi)$ is the ℓ^1-rapid decay algebra associated to π. The groups $\mathcal{P}H^*(\pi;V)$ and thus the properties $\mathcal{P}$-IC and $\mathcal{P}$-SIC are, in general, sensitive to the choice of word-length function on π, and are more precisely functors of (π, L) (unless otherwise indicated, L is assumed to be the stand word-length functon on π). Combined with the corollary above, we have

Theorem B *SNC(π) holds true whenever π is $\mathcal{P}$-IC. In particular, it holds for all $\mathcal{P}$-SIC groups.*

By results of the author and Ji-Ramsey, the class of $\mathcal{P}$-SIC groups is an easily-describable subset of the class of HF^∞ groups (those where $B\pi \simeq X$ a complex with finitely many cells in each dimension). The following theorem represents joint work with R. Ji and B. Ramsey.

Theorem C [JOR2] *The class of $\mathcal{P}$-SIC groups is equal to the class of HF^∞ groups with polynomially bounded Dehn functions in each degree. It includes all discrete groups which are asynchronously combable in polynomial time, and is closed under arbitrary extensions in the category of groups with word-length*[1]*; moreover, if π acts cofinitely and without inversion on a weighted acyclic simplicial complex X whose simplicial chains admit a polynomially bounded chain contraction, and the isotropy group of each simplex is $\mathcal{P}$-SIC with respect to the induced word-length, then π is $\mathcal{P}$-SIC.*

The constraint imposed by the slightly technical "Dehn functions" appearing in the above result is clarified by the following topological reformulation, due to Ji and Ramsey.

Equivalence of Dehn functions - [JR] *For an HF^∞ group π, the Dehn functions of π are polynomially equivalent to the "course" Dehn functions of π, defined using filling norms.*

[1] a short-exact sequence of groups with word-length $(G_1, L_1) \rightarrowtail (G_2, L_2) \twoheadrightarrow (G_3, L_3)$ consists of a short-exact sequence of groups where L_1 is the restriction of L_2 to G_1, and L_3 is the word-length function on G_3 induced by L_2 and the projection $G_2 \twoheadrightarrow G_3$. In particular if G_1 is $\mathcal{P}$-SIC with respect to the standard word-length, then in order for (G_1, L_1) to be $\mathcal{P}$-SIC, one typically needs the image of G_1 to be at most polynomially distorted in G_2 under the injection $G_1 \rightarrowtail G_2$.

In other words, π has polynomially bounded Dehn functions iff there exists an HF^∞ $K(\pi,1)$-complex X whose universal cover $\widetilde{X}$ satisfies the condition:

- *For all $n \geq 1$ there exists a polynomial p_n such that for every map $f : S^n \to \widetilde{X}$ of a combinatorial n-sphere S^n to $\widetilde{X}$, f extends to a map $\tilde{f} : B^{n+1} \to \widetilde{X}$ where B^{n+1} is a combinatorial $n+1$-disk for which (# of cells of B^{n+1}) $\leq p_n$(# cells of S^n).*

The last item in Theorem C generalizes to

Theorem D [JOR2] *If π acts cofinitely and without inversion on a weighted acyclic simplicial complex X whose simplicial chains admit a polynomially bounded chain contraction, and the isotropy group of each simplex is $\mathcal{P}$-IC with respect to the induced word-length, then π is $\mathcal{P}$-IC.*

$\mathcal{P}$-SIC includes all CAT(0)-groups, as such groups admit synchronous linear combings. Poincaré Duality groups provide another class of groups where this condition holds under seemingly mild constraints.

Theorem E [JOR2] *Suppose $B\pi \simeq M$, where M is a compact, closed, oriented n-dimensional manifold. Let $\mu''_\pi \in H^n(M \times M) \cong H^n(\pi \times \pi)$ denote the fundamental cohomology class dual to the diagonal embedding $\Delta(M) \subset M \times M$. If μ''_π is $\mathcal{P}$-bounded, then π is $\mathcal{P}$-IC.*

Conclusion *The groups described by the Theorems C, D, and E all satisfy SNC(π).*

On the other hand, there exist elementary amenable groups which are not $\mathcal{P}$-IC, including solvable groups with quadratic first Dehn function. A detailed discussion of these issues and examples (in both the positive and negative direction) appears in [JOR2].

Cones of effective two-cycles on toric manifolds.

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The Kleiman-Mori cone, that is, the cone of effective one-cycles on an algebraic variety is one of the most important objects in the birational geometry. As a next step, in this talk, we study the cone of effective two-cycles on a smooth projective toric variety. We explain about the combinatorial description for numerical two-cycles on a toric manifold. As an application, we can determine whether a given toric manifold is a 2-Fano manifold or not easily.

The topology of Positive scalar curvature.

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Which compact smooth manifolds (no boundary) do admit a Riemannian metric with positive scalar curvature? If there is one, how many are there?

In this talk, we focus on the second question: does the space of metrics of positive scalar curvature posses a rich topology?

It is well know that the space of such metrics, as well as the moduli space (moduli the action of the diffeomorphism group) has typically infinitely many components; detected by index invariants or secondary index invariants (rho invariants) (results of Hitchin, Lawson,..) Moreover, Hitchin shows that in certain dimensions the first homotopy group of the space of metrics is non-trivial. Some people conjectured, that all components of the moduli space are contractible.

In joint work with Boris Botvinnik, Bernhard Hanke, Mark Walsh we use the topology of the classifying space of the Diffeomorphism group of D^N (in particular non-trivial higher torsion) to produce non-trivial homotopy classes in π_n for arbitrarily large n. The construction is based on a famous smooth fiber bundle by Hatcher. However, these classes do not lift to the space of metrics itself.

In joint work with Diarmuid Crowley we show that also the space of metrics has non-trivial classes in π_n for arbitrarily large n. They are transported from Diff via the action and are based on non-trivial products in stable homotopy groups, detected in real K-theory.

We will explain the relevant invariants and the constructions of the families of metrics.

Manifolds with torus action

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Suppose that k-dimensional torus T^k semi-free act on n-dimensional closed smooth manifold $W^n (n > k \geqslant 1)$ and let M_1^l,..., M_p^l closed submanifolds of fixed points. Let $f : W^n \to \mathbb{R}$ be a smooth T^k-invariant function on W^n and let Σ_f the set of singular points of f.

Definition 1. The function f is called Morse-Bott function if Σ_f is disjont union of non-degenerate closed submanifolds M_1^l,..., M_p^l and and some number of T^k.

Theorem 1. *Every smooth semi-free T^k action on manifold W^n with M_1^l,..., M_p^l submanifolds of fixed points has an T^k-invariant Morse-Bott function f.*

Definition 2. Let f be an T^k-invariant Morse-Bott function for smooth semi-free T^k action with M_1^l,..., M_p^l submanifolds of fixed points on W^n. Suppose that the index of a critical submanifold M_i^l of f is λ_i. **The state of** f is the collection of numbers $\lambda_1, \lambda_2, \ldots, \lambda_p$, which we will be denoted by $St_f(\lambda_i)$.

Definition 3. Let W^n be a manifold with smooth semi-free T^k-action which has M_1^l,..., M_p^l submanifolds of fixed points. The T^k-**equivariant Morse number** $\mathcal{M}^{\nu}_{T^k}(W^n, St(\lambda_i))$ **of index** ν **of a state** $St(\lambda_i)$ of W^n is the minimum number of singular T^k of index ν taken over all T^k-invariant Morse-Bott functions on W^n with state $St(\lambda_i)$. The T^k-**equivariant Morse number** $\mathcal{M}^{\nu}_{T^k}(W^n)$ of index ν of W^n is the minimum number of $\mathcal{M}^{\nu}_{T^k}(W^n, St(\lambda_i))$ taken over all states. The T^k-**equivariant Morse number** $\mathcal{M}_{T^k}(W^n, St(\lambda_i))$ **of a state** $St(\lambda_i)$ is the minimum number of singular T^k of all indices taken over all T^k-invariant Morse-Bott functions on W^n with state $St(\lambda_i)$. The T^k-**equivariant Morse number** $\mathcal{M}_{T^k}(W^n)$ of W^n is the minimum number of $\mathcal{M}_{T^k}(W^n, St(\lambda_i))$ taken over all states.

There is an unsolved problem: for a manifold W^n with a semi-free T^k-action which has M_1^l,..., M_p^l submanifolds of fixed points **find exact values of** $\mathcal{M}^{\nu}_{T^k}(W^n, St(\lambda_i))$**,** $\mathcal{M}^{\nu}_{T^k}(W^n)$**,** $\mathcal{M}_{T^k}(W^n, St(\lambda_i))$**, and** $\mathcal{M}_{T^k}(W^n)$.

Definition 4. An T^k-invariant Morse-Bott function f on W^n with semi-free T^k-action which has which has M_1^l,..., M_p^l submanifolds of fixed points is

- **minimal for index** ν **of a state** $St(\lambda_i)$ if the number of singular T^k of f of index ν is equal to $\mathcal{M}^{\nu}_{T^k}(W^n, St(\lambda_i))$;
- **minimal for index** ν if the number of singular T^k of f of index ν is equal to $\mathcal{M}^{\nu}_{T^k}(W^n)$;
- **minimal for state** $St(\lambda_i)$ if the number of all singular circles of f is equal to $\mathcal{M}_{T^k}(W^n, St(\lambda_i))$;
- **minimal** if the number of all singular T^k of f is equal to $\mathcal{M}_{T^k}(W^n)$.

Theorem 2. *Let W^n $(n > 2k)$ be a simply-connected manifold with free homology group of $H_i(W^n, \mathbb{Z})$ admits a smooth semi-free T^k-action which has M_1^l,..., M_p^l submanifolds of fixed points. Then on W^n for the sequence $(0, \ldots, 0, n-l, \ldots, n-l)$ there exists a minimal T^k-invariant Morse-Bott function g for the state $St(0, \ldots, 0, n-l, \ldots, n-l)$.*

Characteristic classes of simplicial manifolds: combinatorics, electric circuits and homological algebra.

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This talk is a report on the ongoing joint project of the author and Nikolai Mnev from PDMI.

In topology the term "simplicial n-manifold" refers to a n-dimensional simplicial complex, such that the links of all its k-simplices for all k are simplicial subdivisions of the standard $n-k-1$-dimensional sphere. It is well-known, that one can define characteristic classes (Pontrjagin classes, Euler class, etc.) for arbitrary simplicial manifolds, or, more generally, for any homological manifold (i.e. where the condition above is replaced by a still weaker condition, that all the links have homology, isomorphic to that of the corresponding sphere), in such a way that the expressions depend only on the local combinatorial structure of the complex (this was proved for example by R.Thom, Levitt and Rourke etc.). The problem of finding such expressions in an explicit form is a long-standing one. Among many authors that contribute to this field one can list Levitt and Rourke, Gabrielov, Gelfand and Losik, McPherson, Gaifullin etc.

In my talk I am going to give a description of a new approach to this question, which is based on the extensive use of the recently discovered by N.Mnev combinatoric construction, equivalent to the PL bundles. This construction (called the "combinatorial sphere bundle" or the "Gauss functor") associates to every simplex in a simplicial manifold X a cellular subdivision of $n$sphere and an aggregation map to every pair of adjoint simplices in X. It was shown by N.Mnev that this data is equivalent to the "normal PL bundle of the diagonal embedding" used by Levitt and Rourke.

In our joint work we use this construction and pass from the geometric picture to the level of cochain complexes. In order to associate a chain map to an (abstract) aggregation we resort to the canonical euclidean structures on the complexes. This approach is quite canonical and enables one to write down explicit formulas for some characteristic classes, e.g. the Euler class. Calculations thereof involves manipulations with the so-called Kirhgoff laws for calculation of electric flows in an electric circuits and give quite unexpected coefficients. On the other hand, once in the domain of homology one can use the standard tools of the homological algebra, such as the homotopy equivalence of the DG algebras, perturbation technics etc., which leads one to a rather involved, but quite explicit formulas, that give an analog of the noncommutative 1-cocycle, twisting cochains with values in an appropriate Cech complex and finally enables one to mimick the constructions of Bott, Dupont and the author of the characteristic classes.

Classification of embeddings below the metastable dimension.

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Statement of the main result.

For a smooth (piecewise linear, PL) manifold N denote by $E^m(N)$ ($E^m_{PL}(N)$) the set of smooth (PL) embeddings $N \to \mathbb{R}^m$ up to smooth (PL) isotopy.

The 'connected sum' commutative group structure on $E^m(S^q)$ was defined for $m \geq q+3$ by Haefliger. We define an 'S^p-parametric connected sum' commutative group structure on $E^m(S^p \times S^q)$ and $E^m_{PL}(S^p \times S^q)$ for $m \geq 2p+q+3$; cf. [Sk07, PCS].

Denote by $V_{k,l}$ the Stiefel manifold of l-frames in $\mathbb{R}^k$.

Main Theorem. [Sk06] *For $p \geq 1$ and $m \geq \max\{2p+q+3, \frac{2p+3q+3}{2}\}$*

$$E^m(S^p \times S^q) \cong \pi_q(V_{m-q,p+1}) \oplus E^m(S^{p+q}) \quad \textit{and} \quad E^m_{PL}(S^p \times S^q) \cong \pi_q(V_{m-q,p+1})/s_{m,p,q},$$

where for $2m > 2p+3q+3$ we have $s_{m,p,q} = 0$ while for $2m = 2p+3q+3$ we set $l := m-p-q-1 = (q+1)/2$ and we have that $s_{m,p,q}$ is an integer multiple of the image of $[\iota_l, \iota_l]$ under the map μ'' from the exact sequence $\pi_q(S^l) \xrightarrow{\mu''} \pi_q(V_{m-q,p+1}) \to \pi_q(V_{m-q,p})$ of the bundle defined by forgetting the last vector.

In the smooth case the isomorphism from the right to the left is $\tau \oplus \#$. Here $\#(g)$ is the connected sum of the standard embedding $S^p \times S^q \to \mathbb{R}^m$ with embedding $g : S^{p+q} \to \mathbb{R}^m$. The PL analogue τ_{PL} is defined analogously and is an epimorphism with the kernel $s_{m,p,q}$.

Definition of the map τ [Sk02, proof of Torus Lemma 6.1, Sk08, §6]. Recall that $\pi_q(V_{m-q,p+1})$ is isomorphic to the group of smooth maps $S^q \to V_{m-q,p+1}$ up to smooth homotopy. These maps can be considered as smooth maps $\varphi : S^q \times S^p \to \partial D^{m-q}$. Define the smooth embedding $\tau(\varphi)$ as the composition

$$S^p \times S^q \xrightarrow{\varphi \times \mathrm{pr}_2} \partial D^{m-q} \times S^q \subset D^{m-q} \times S^q \subset \mathbb{R}^m.$$

Here pr_2 is the projection onto the second factor and $\subset$ are standard inclusions.

Discussion of the main result.

This paper is on the classical Knotting Problem: *for an n-manifold N and a number m, describe isotopy classes of embeddings $N \to \mathbb{R}^m$*. For recent surveys see [RS99, Sk08, HCEC].

Many interesting examples of embeddings are embeddings $S^p \times S^q \to \mathbb{R}^m$, i.e. *knotted tori*. See references in [KT]. A classification of knotted tori is a natural next step (after the Haefliger link theory and the classification of embeddings of highly-connected manifolds) towards classification of embeddings of *arbitrary* manifolds. Since the general Knotting Problem is very hard [HCEC], it is very interesting to solve it for the important particular case of knotted tori. Classification results for knotted tori gives some insight or even precise information concerning arbitrary manifolds (this is formalized in [Sk07], [Sk10], [PCS]) and reveals new interesting relations to algebraic topology.

We have $E^m(S^p \times S^q) = E^m_{PL}(S^p \times S^q) = 0$ for $p \leq q$ and $m \geq p+2q+2$ [Sk08, Theorem 2.8.b]. In particular, the Main Theorem is trivial for $p \geq q$. From now on assume that $p < q$.

The Knotting Problem is more accessible for

$$2m \geq 3n + 4.$$

In particular, $E^m(S^n) = 0$ for $2m \geq 3n + 4$. Thus for $2m \geq 3p + 3q + 4$ the Main Theorem is known [Sk02, Corollary 1.5].

The Knotting Problem is much harder for $2m < 3n+4$: if N is a closed manifold that is not a disjoint union of spheres, then until recent results no complete readily calculable smooth isotopy classification was known, in spite of the existence of the interesting approaches of Browder-Wall and Goodwillie-Weiss.

However, if a closed manifold N is d-connected and

$$3n + 4 > 2m \geq 3n + 3 - d,$$

there are classification results in the PL category [Sk02, Sk08, §3]. Thus the PL case of the Main Theorem is known for $3p + 3q + 4 > 2m \geq 2p + 3q + 4$. *The smooth case of the Main Theorem for $3p + 3q + 4 > 2m \geq 2p + 3q + 4$ is new but is not hard.* This case follows by the PL case and the following result [Sk06]:

$$E^m(S^p \times S^q) \cong \frac{E^m(S^p \times S^q)}{\operatorname{im} \#} \oplus E^m(S^{p+q}) \quad \text{for} \quad m \geq 2p + q + 3.$$

If N is a closed d-connected n-manifold, then the Knotting Problem is much harder for $2m \leq 3n+2-d$ because even the classification in the PL category is unknown (and methods of [Sk02, Sk08', Sk05, CS08] do not work without modification). *The most interesting and the most diffi cult case of the Main Theorem is $2m = 2p + 3q + 3$,* which corresponds to the 'boundary' case

$$2m = 3n + 2 - d.$$

This case $2m = 3p+2q+3$ requires new ideas which are hopefully are interesting in themselves. The methods which we develop for the 'boundary' case can be extended to classify embeddings in other cases [CRS07, CRS08].

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Toric genera of homogeneous spaces and their fibrations.

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We present the results on universal toric genus obtained as continuation of our work [3] on universal toric genus of homogeneous spaces.

The notion of universal toric genus was introduced in [2] and explained in detail in [1]. It can be constructed for any even dimensional manifold M^{2n} with a given torus action and stable complex structure which is equivariant under the torus action. If the torus action has finite set of isolated fixed points than the universal toric genus for such action can be localized meaning that it can be expressed in terms of signs and weights at fixed points for the representations that gives arise from the given torus action. Appealing to this result, it is obtained in [3] an explicit formula for the universal toric genus of homogeneous spaces G/H endowed with the canonical action of the common maximal torus for G and H, where $\operatorname{rk} H = \operatorname{rk} G$. Applying the Chern-Dold character, the explicit formulas for the complex cobordism classes as well as the Chern characteristic numbers for these spaces are established.

We study further the notion of the universal toric genus on some specific homogeneous spaces and expand it to some fibrations. We generalize our results from [3] on universal toric genus of flag and Grassman manifolds to an arbitrary invariant almost complex structure as well as to the generalized Grassman manifolds. We also consider homogeneous fibrations $H/K \to G/K \to G/H$ where all groups have equal ranks and assume that the fiber and the base are endowed with the invariant almost complex structures. It is obtained the explicit formula, in terms of the local data for the base and the fiber, for the universal toric genus as well as for the cobordsim class for G/K related to the induced invariant almost complex structure. It is also studied the muiltiplicativity problem for universal toric genus of these fibrations and established the effective criterion for it to be classically multiplicative. We expand the results from homogeneous to an arbitrary fibrations. Given two stable complex manifolds X and F with an equivariant action of the torus T^k we provide the construction of a T^k- equivariant stable complex manifold E which fibers over X with fiber F and whose stable complex structure and torus action naturally arises from those of X and F. The explicit formula for the universal toric genus of E is obtained and established the condition to be multiplicative. As an application we provide, in particular, some characteristic fibrations of compex and quaternionic flag manifolds $U(n)/T^n$ and $Sp(n)/T^n$.

The talk is based on the joint work with Victor M. Buchstaber [4].

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Homotopy decompositions of gauge groups and applications to moduli spaces.

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We focus on gauge groups of principal U(n)-bundles over orientable Riemann surfaces. In many cases the gauge groups are decomposed, up to homotopy equivalence, as products of other more well known spaces. As a consequence, the calculation of their homotopy groups reduces to that of spheres or U(n).This can then be applied to calculate the homotopy groups of certain moduli spaces through a range, answering a question of Daskalopoulos and Uhlenbeck.

Complex structures on moment-angle-manifolds.

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Moment-angle-complexes $\mathcal{Z}_K$, associated to a simplicial complexes K, were studied by many authors in context of 'toric topology'. It is well-known, that in many cases moment-angle-complexes are topologic manifolds. We show that the moment-angle-manifold associated to a complete simplcial fan admits smooth and complex structures. Moment-angle-manifolds endowed with these complex structures provide an interesting series of compact non-symplectic complex manifolds, including classic examples of Hopf and Calabi-Eckmann. We describe these manifolds as holomorphic principal T^k-bundles over compact toric varieties and compute (additively) their Dolbeault cohomology.

Gaudin subalgebras and stable rational curves

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Gaudin subalgebras are abelian Lie subalgebras of maximal dimension spanned by generators of the Kohno–Drinfeld Lie algebra t_n, which can be interpreted as the holonomy Lie algebra of the configuration space of n distinct points on the complex plane, or the set of values for the universal Knizhnik-Zamolodchikov connection.

I will explain that Gaudin subalgebras form a variety isomorphic to the moduli space $\bar{M}_{0,n+1}$ of stable curves of genus zero with $n+1$ marked points. In particular, this gives an embedding of $\bar{M}_{0,n+1}$ in a Grassmannian of $(n-1)$-planes in an $n(n-1)/2$-dimensional space.

The talk is based on a joint work with Aguirre and Felder.

Upper and lower bounds for nestohedra

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Simple polytopes play important role in toric geometry and topology. The classical problem of upper and lower bounds for h-vectors of n-dimensional simple polytopes with fixed number of facets is solved in [Ba1], [Ba2] and [Mc].

Nowadays there appeared an important subclass of simple polytopes - Delzant polytopes. For every Delzant polytope P^n there exists a Hamiltonian toric manifold M^{2n} such that P^n is the image of the moment map. Davis-Januszkiewicz theorem states that odd Betti numbers $b_{2i-1}(M^{2n})$ are zero and even Betti numbers $b_{2i}(M^{2n})$ are equal to components $h_i(P^n)$ of the h-vector of P^n. So, the problem of upper and lower bounds for h-vectors of Delzant polytopes become actual, because its solution gives upper and lower bounds for Betti numbers of Hamiltonian toric manifolds.

Feichtner and Sturmfels (see [FS]) and Postnikov (see [1]) showed that the Minkowski sum of some set of regular simplices is a simple polytope if this set satisfies certain combinatorial conditions identifying it as a buiding set. The resulting family of simple polytopes was called nestohedra in [PRW] because of their connection to nested sets considered by De Concini and Procesi in the context of subspace arrangements. From results of [FS] directly follows that nestohedra are Delzant polytopes. Special cases of building sets are vertex sets of connected subgraphs in a given graph; the corresponding nestohedra called graph-associahedra by Carr and Devadoss were first studied in [CD].

From [FM] we know that if $B_1 \subseteq B_2$ for connected building sets, then P_{B_2} is obtained from P_{B_1} by sequential shaving some faces, consequently, $h_i(P_{B_1}) \leqslant h_i(P_{B_2})$. Therefore, $h_i(\Delta^n) \leqslant h_i(P_B) \leqslant h_i(Pe^n)$ for every n-dimensional nestohedron P_B and these bounds are unimprovable.

In the combinatorics of flag simple polytopes especially interested is γ-vector. Using [Bu1] and definitions of g-,h- and f-vectors one can prove that componentwise inequality $\gamma(P_1) \leqslant \gamma(P_2)$ for simple n-polytopes P_1 and P_2 implies componentwise inequalities:$g(P_1) \leqslant g(P_2), h(P_1) \leqslant h(P_2), f(P_1) \leqslant f(P_2)$.

Gal's conjecture (which is a generalization of famous Charney-Davis conjecture) states that flag simple polytopes have nonnegative γ-vectors (see [G]). In [Bu2] it was described realization of the associahedron as a polytope obtained from the standard cube by shaving faces of codimension 2. We show that every flag nestohedron has such a realization. As a corollary we derive that unimprovable bounds for γ-vectors of flag nestohedra are $\gamma(I^n)$ and $\gamma(Pe^n)$. That includes Gal's conjecture for flag nestohedra, since $\gamma_i(I^n) = 0, i > 0$.

There are remarkable series of graph-associahedra corresponding to series of graphs: associahedra As^n (path graphs), cyclohedra Cy^n (cyclic graphs), permutohedra Pe^n (complete graphs) and stellohedra St^n (star graphs). Using these series we obtain upper and lower bounds for γ-vectors of graph-associahedra and some its important subclasses.

The main result is following:

Theorem. *There are following unimprovable bounds:*

1) $\gamma_i(I^n) \leqslant \gamma_i(P_B) \leqslant \gamma_i(Pe^n)$ *for any flag n-dimensional nestohedron P_B;*

2) $\gamma_i(As^n) \leqslant \gamma_i(P_{\Gamma_{n+1}}) \leqslant \gamma_i(Pe^n)$ *for any connected graph Γ_{n+1} on $[n+1]$;*

3) $\gamma(Cy^n) \leqslant \gamma_i(P_{\Gamma_{n+1}}) \leqslant \gamma_i(Pe^n)$ *for any Hamiltonian graph Γ_{n+1} on $[n+1]$;*

4) $\gamma_i(As^n) \leqslant \gamma_i(P_{\Gamma_{n+1}}) \leqslant \gamma_i(St^n)$ for any tree Γ_{n+1} on $[n+1]$.

The similar bounds also hold for f-,g- and h-vectors.

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Torus fibrations and localization of index.

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This talk is based on the joint work [3, 4] with Hajime Fujita and Mikio Furuta.

Let M be a possibly non-compact Riemannian manifold and W a $\mathbb{Z}_2$-graded Clifford module bundle on M. Suppose that M has an open subset V with complement $M \setminus V$ compact and V is covered by finitely many open subsets $\{V_\alpha\}_{\alpha \in A}$ such that each V_α has a structure of the total space of a torus bundle $\pi_\alpha \colon V_\alpha \to U_\alpha$ and each $W|_{V_\alpha}$ is equipped with a Dirac-type operator D_α along fibers of π_α. Under some compatibility and acyclicity conditions we will show that there exists an integer $\mathrm{ind}(M, V)$ depending on the all data such that $\mathrm{ind}(M, V)$ has the following properties:

1. $\mathrm{ind}(M, V)$ is invariant under continuous deformations of the data.

2. If M is closed, then $\mathrm{ind}(M, V)$ is equal to the index $\mathrm{ind}\, D$ of a Dirac-type operator D on W.

3. Suppose V' is an open subset of V with $M \setminus V'$ compact such that the torus bundle structures on V can be restricted to V'. Then we have
$$\mathrm{ind}(M, V) = \mathrm{ind}(M, V').$$

4. Suppose M' is an open neighborhood of $M \setminus V$ such that the torus bundle structures on V can be restricted to $V \cap M'$. Then $\mathrm{ind}(M, V)$ has the following excision property
$$\mathrm{ind}(M, V) = \mathrm{ind}(M', V \cap M').$$

5. Suppose M is a disjoint union $M = M_1 \coprod M_2$. Then we have the following sum formula
$$\mathrm{ind}(M, V) = \mathrm{ind}(M_1, V \cap M_1) + \mathrm{ind}(M_2, V \cap M_2).$$

6. We have a product formula for $\mathrm{ind}(M, V)$. For the precise statement see [4, Theorem 5.8].

We call $\mathrm{ind}(M, V)$ a *local index*. In the case where M is closed, as a corollary we obtain a localization formula for the index of a Dirac-type operator on W.

To construct $\mathrm{ind}(M, V)$ we introduce a deformation of a Dirac-type operator by using D_α's. The deformation allows an interpretation as an adiabatic limit or an infinite dimensional analogue of Witten's deformation.

We will describe an application to symplectic geometry. For a closed symplectic manifold with prequantization line bundle the *Riemann-Roch number* is defined to be the index of a Spin^c Dirac operator with coefficients in the prequantization line bundle.

Suppose the symplectic manifold is equipped with a structure of the total space of a Lagrangian fiber bundle. Note that the restriction of the prequantization line bundle to each fiber is flat. A fiber of the Lagrangian fiber bundle is said to be *Bohr-Sommerfeld* if the restriction of the prequantization line bundle to the fiber is trivially flat. Bohr-Sommerfeld

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fibers appear discretely. Then it is known in [1] that the Riemann-Roch number is equal to the number of Bohr-Sommerfeld fibers.

Similar phenomena have been observed for several examples of Lagrangian fiber bundles with singular fibers, such as,

- moment maps of toric varieties [2],
- Gelfand-Cetlin's completely Hamiltonian system for a complex flag manifold [7],
- Goldman's Hamiltonian system on the moduli space of flat SU(2)-bundle on a Riemann surface [5, 9],

and for non symplectic generalizations

- presymplectic toric manifolds [10],
- Spin^c manifolds [6],
- torus manifolds [11, 8],

and so on.

We will apply the localization formula for the local index to understand these phenomena and show that the Riemann-Roch number is described as the sum of the number of nonsingular Bohr-Sommerfeld fibers and the contributions from singular fibers.

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Discrete and continuous complexes and posets in topological combinatorics.

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We collect together and analyze from the same point of view some important classes of complexes which exhibit both discrete and continuous nature and which have continually played an important role in topology, combinatorics and their applications. Among the key examples are:

(1) Moment-angle complexes $\mathcal{Z}_K$ and polyhedral product functors $(X, A)^K$ as "continuous complexes" over a discrete (simplicial) complex K.

(2) Homotopy colimits of diagrams as continuous complexes over the corresponding (face) poset.

(3) Continuous posets in the sense of Vassiliev and discrete-continuous polytopes in the sense of Kalai and Wigderson, continuous "Bier-spheres".

(4) Convex hulls of neighborly embedded manifolds (complexes).

(5) "Continuous inflation" of simplicial (polyhedral) complexes as a continuous analog of the discrete inflation of complexes (A. Björner, M. Wachs, V. Welker, *Poset Fiber Theorems*, T.A.M.S., 2005).

(6) Complexes of "vertex-colored polytopes" (after de Longueville and Živaljević) with discrete and continuous sets of colors, etc.

A unified framework for studying discrete-continuous analogues of complexes and posets (with the emphasis on combinatorially and geometrically motivated constructions and invariants) was proposed by the author in the paper "Combinatorics of topological posets: Homotopy complementation formulas", *Adv. Appl. Math.* 21 (1998), 547Џ-574.

This paper continued the program of using homotopy colimits and related constructions in geometric and topological combinatorics which was initiated in Ziegler-Živaljević, *Math. Ann.* 1993, and Welker-Ziegler-Živaljević, *J. Reine Angew. Math.* 1999. For more recent applications of homotopy colimits and other homotopical methods the reader is referred to Panov-Ray-Vogt, arXiv:math/0202081v1 [math.AT]; Panov-Ray, arXiv:0707.0300v2 [math.AT], and Bahri-Bendersky-Cohen-Gitler, arXiv:1001.3372v1 [math.AT].

We plan to give a brief overview of the area emphasizing the interplay of discrete and continuous in some fundamental constructions (Vassiliev geometric resolutions, convex hulls of neighborly polytopes, etc.). As an illustration of the use of complexes of "vertex-colored polytopes" (joint work with Mark de Longueville, *Advances in Mathematics*, 2008) we exhibit a "Multidimensional splitting necklace theorem" which extends the well known one-dimensional case due to Noga Alon.

Section "Algebra and Number Theory"

On symmetry groups of quasicrystals.

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A mathematical model for crystals was presented by B. Delonias. It involves the notion of a symmetry group of a crystal as a subgroup of isometry group of an Euclidean space.

In 1984 a new alloy $Al_{0,86}Mn_{0,14}$ was discovered with a symmetry which was not allowed in the symmetry theory of crystals. The new metallic alloys are called quasicrystals.

The most common mathematical models of quasicrystals is a *cut and project scheme.* Let E be an Euclidean space with a direct decomposition $E = U \oplus V$ and with a discrete subgroup M such that E/M is compact, $M \cap V = 0$ and $\rho(M)$ is dense in V. Consider the diagram of projections of groups

$$\begin{array}{ccccc} U & \xleftarrow[\pi]{} & E & \xrightarrow{\rho} & V\ . \\ & & \cup & & \\ & & M & & \end{array}$$

A nonempty compact convex subset $W \subset V$ is a *window* if W is the completion of its interior. A *proper symmetry group* $Sym_W Q$ of a quasicrystal Q is the group of all affine transformations of the hyperspace E which map the set Q bijectively onto itself. A *general symmetry group* Sym is a group of all affine transformation of the hyperspace E such that M is Sym-invariant and U is stable under all differentials of elements of Sym. It is shown that $\mathrm{Sym}_W Q \subseteq \mathrm{Sym}$.

These is found a classification of subgroups of Sym of the form $\mathrm{Sym}_W Q$ for some window W. The class of these subgroups contain finite subgroups of Sym. There is found a classification of finite subgroups in Sym in the case when dimensions of U, V is at most 3.

Simultaneous Diophantine approximations and generalizations of the continued fraction.

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In the space $\mathbb{R}^n$, suppose that we are given l homogeneous linear forms and k homogeneous quadratic forms; each quadratic form is the product of two complex conjugate linear forms, and $l + 2k = n$. The moduli of all m forms $(m = k + l)$ define a mapping of $\mathbb{R}^n$ to the nonnegative orthant $\mathbb{R}^m_+$ of the space $\mathbb{R}^m$. Nonzero integer points from $\mathbb{R}^n$ are mapped to a set $\mathbf{Z} \subset \mathbb{R}^m_+$. The closure of the convex hull $\mathbf{G}$ of $\mathbf{Z}$ is a polyhedral set in $\mathbb{R}^m_+$. Its boundary $\partial\mathbf{G}$ is of dimension $m - 1$ and contains the images of the best Diophantine approximations to the root subspaces of all m forms. In the algebraic case, m forms are related in a certain manner to the roots of an irreducible polynomial of degree n that has l real roots and k pairs of complex conjugate roots. It is proved that, in the algebraic case, the boundary $\partial\mathbf{G}$ has $m-1$ independent periods. This is a generalization of Lagrange's theorem on the periodicity of the continued fraction of a quadratic irrationality.

For small m see [1–6] and for arbitrary m see [7].

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Transference inequalities for Diophantine exponents.

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Consider a system of linear equations

$$\Theta\mathbf{x} = \mathbf{y}$$

with $\mathbf{x} \in \mathbb{R}^m$, $\mathbf{y} \in \mathbb{R}^n$ and

$$\Theta = \begin{pmatrix} \theta_{11} & \cdots & \theta_{1m} \\ \vdots & \ddots & \vdots \\ \theta_{n1} & \cdots & \theta_{nm} \end{pmatrix}, \qquad \theta_{ij} \in \mathbb{R}.$$

Definition 7. The supremum of real numbers γ, such that there are infinitely many $\mathbf{x} \in \mathbb{Z}^m$, $\mathbf{y} \in \mathbb{Z}^n$ satisfying the inequality

$$|\Theta\mathbf{x} - \mathbf{y}|_\infty \leqslant |\mathbf{x}|_\infty^{-\gamma},$$

where $|\cdot|_\infty$ denotes the sup-norm in the corresponding space, is called the *individual Diophantine exponent* of Θ and is denoted by $\beta(\Theta)$.

Definition 8. The supremum of real numbers γ, such that for each t large enough there are $\mathbf{x} \in \mathbb{Z}^m$, $\mathbf{y} \in \mathbb{Z}^n$ satisfying the inequalities

$$0 < |\mathbf{x}|_\infty \leqslant t, \qquad |\Theta\mathbf{x} - \mathbf{y}|_\infty \leqslant t^{-\gamma},$$

is called the *uniform Diophantine exponent* of Θ and is denoted by $\alpha(\Theta)$.

The talk is devoted to the relations between the quantities $\alpha(\Theta)$, $\alpha(\Theta^\intercal)$, $\beta(\Theta)$, $\beta(\Theta^\intercal)$, where $\Theta^\intercal$ denotes the transposed matrix. New inequalities will be proposed, which generalize or refine the existing results of Jarn´ik, Khintchine, Apfelbeck, Dyson, Laurent, Bugeaud and others. Besides that, the method used to obtain these inequalities allowed to improve the classical Mahler's transference theorem.

1. Uniform exponents

The strongest result connecting $\alpha(\Theta)$ and $\alpha(\Theta^\intercal)$ used to be the following theorem proved by Apfelbeck:

Theorem 1. (i) *We always have*

$$\alpha(\Theta^\intercal) \geqslant \frac{n\alpha(\Theta) + n - 1}{(m-1)\alpha(\Theta) + m}.$$

(ii) *If $m > 1$ and $\alpha(\Theta) > (2(m+n-1)(m+n-3)+m)/n$, then*

$$\alpha(\Theta^\intercal) \geqslant \frac{1}{m}\left(n + \frac{n(n\alpha(\Theta) - m) - 2n(m+n-3)}{(m-1)(n\alpha(\Theta)-m) + m - (m-2)(m+n-3)}\right).$$

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Our first result improves Theorem 1:

Theorem 1. *For all positive integers n, m, not equal simultaneously to 1, we have*

$$\alpha(\Theta^\intercal) \geqslant \begin{cases} \dfrac{n-1}{m-\alpha(\Theta)}, & \text{if } \alpha(\Theta) \leqslant 1, \\[2mm] \dfrac{n-\alpha(\Theta)^{-1}}{m-1}, & \text{if } \alpha(\Theta) \geqslant 1. \end{cases}$$

2. Individual exponents

The classical Khintchine's transference theorem connects $\beta(\Theta)$ and $\beta(\Theta^\intercal)$ in the case $n = 1$:

Theorem 2. *If $n = 1$, then*

$$\frac{\beta(\Theta)}{(m-1)\beta(\Theta)+m} \leqslant \beta(\Theta^\intercal) \leqslant \frac{\beta(\Theta)-m+1}{m}.$$

These inequalities cannot be improved if only $\beta(\Theta)$ and $\beta(\Theta^\intercal)$ are considered. However, stronger inequalities can be obtained if $\alpha(\Theta)$ and $\alpha(\Theta^\intercal)$ are also taken into account. The corresponding result belongs to Laurent and Bugeaud:

Theorem 3. *If $n = 1$, then*

$$\frac{(\alpha(\Theta)-1)\beta(\Theta)}{((m-2)\alpha(\Theta)+1)\beta(\Theta)+(m-1)\alpha(\Theta)} \leqslant \beta(\Theta^\intercal) \leqslant \frac{(1-\alpha(\Theta^\intercal))\beta(\Theta)-m+2-\alpha(\Theta^\intercal)}{m-1}.$$

Theorem 2 was later generalized to the case of arbitrary n, m by Dyson (a simpler proof was later obtained by Khintchine):

Theorem 4. *For all n, m, not equal simultaneously to 1,*

$$\beta(\Theta^\intercal) \geqslant \frac{n\beta(\Theta)+n-1}{(m-1)\beta(\Theta)+m}.$$

Our second result generalizes Theorem 3 and improves Theorem 4 the way Theorem 3 improves Theorem 2:

Theorem 2. *For all positive integers n, m, not equal simultaneously to 1, we have three inequalities*

$$\begin{aligned} \beta(\Theta^\intercal) &\geqslant \frac{n\beta(\Theta)+n-1}{(m-1)\beta(\Theta)+m}, \\ \beta(\Theta^\intercal) &\geqslant \frac{(n-1)(1+\beta(\Theta))-(1-\alpha(\Theta))}{(m-1)(1+\beta(\Theta))+(1-\alpha(\Theta))}, \\ \beta(\Theta^\intercal) &\geqslant \frac{(n-1)(1+\beta(\Theta)^{-1})-(\alpha(\Theta)^{-1}-1)}{(m-1)(1+\beta(\Theta)^{-1})+(\alpha(\Theta)^{-1}-1)}. \end{aligned}$$

3. Transference theorem

One of the strongest theorems describing Khintchine's transference principle belongs to Mahler:

Theorem 5. *If there are* $\mathbf{x} \in \mathbb{Z}^m$, $\mathbf{y} \in \mathbb{Z}^n$, *such that*

$$0 < |\mathbf{x}|_\infty \leqslant X, \qquad |\Theta\mathbf{x} - \mathbf{y}|_\infty \leqslant U,$$

then there are $\mathbf{x} \in \mathbb{Z}^m$, $\mathbf{y} \in \mathbb{Z}^n$, *such that*

$$0 < |\mathbf{y}|_\infty \leqslant Y, \qquad |\Theta^\intercal\mathbf{y} - \mathbf{x}|_\infty \leqslant V,$$

where

$$Y = (d-1)\big(X^m U^{1-m}\big)^{\frac{1}{d-1}}, \quad V = (d-1)\big(X^{1-n} U^n\big)^{\frac{1}{d-1}}, \quad \text{and} \quad d = n+m.$$

Our third result improves Theorem 5. Namely, we substitute the factor $d-1$ by a smaller factor tending to 1 as $d \to \infty$. In order to give the precise statement let us denote by $\mathcal{B}_\infty^d$ the unit ball in the sup-norm in $\mathbb{R}^d$, i.e. the cube

$$\Big\{\mathbf{x} = (x_1, \ldots, x_d) \in \mathbb{R}^d \,\Big|\, |x_i| \leqslant 1,\ i = 1, \ldots, d\Big\}$$

and set

$$\Delta_d = \frac{1}{2^{d-1}\sqrt{d}} \operatorname{vol}_{d-1} \Big\{\mathbf{x} \in \mathcal{B}_\infty^d \,\Big|\, \sum_{i=1}^{d} x_i = 0\Big\},$$

where $\operatorname{vol}_{d-1}(\cdot)$ denotes the $(d-1)$-dimensional Lebesgue measure.

It follows from Vaaler's and Ball's theorems that the volume of each $(d-1)$-dimensional central section of $\mathcal{B}_\infty^d$ is bounded between 2^{d-1} and $2^{d-1}\sqrt{2}$. Hence

$$\sqrt{d/2} \leqslant \Delta_d^{-1} \leqslant \sqrt{d},$$

which implies that $\Delta_d^{-\frac{1}{d-1}} \to 1$ as $d \to \infty$. The following Theorem improves Theorem 5:

Theorem 6. *If there are* $\mathbf{x} \in \mathbb{Z}^m$, $\mathbf{y} \in \mathbb{Z}^n$, *such that*

$$0 < |\mathbf{x}|_\infty \leqslant X, \qquad |\Theta\mathbf{x} - \mathbf{y}|_\infty \leqslant U,$$

then there are $\mathbf{x} \in \mathbb{Z}^m$, $\mathbf{y} \in \mathbb{Z}^n$, *such that*

$$0 < |\mathbf{y}|_\infty \leqslant Y, \qquad |\Theta^\intercal\mathbf{y} - \mathbf{x}|_\infty \leqslant V,$$

where

$$Y = \Delta_d^{-\frac{1}{d-1}}\big(X^m U^{1-m}\big)^{\frac{1}{d-1}}, \quad V = \Delta_d^{-\frac{1}{d-1}}\big(X^{1-n} U^n\big)^{\frac{1}{d-1}}.$$

Parameterized differential Galois theory.

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The talk is based on a common work with A.Ovchinnikov and H.Gillet.

It is known that symmetries of solution spaces of linear differential equations are algebraic groups. Given a family of linear differential equations, one often obtains (non-linear) differential equations with respect to parameters of the base on the matrix elements of the corresponding fiber-wise groups of symmetries. This leads to parameterized differential Galois groups, which are symmetries of solution spaces of linear differential equations with parameters that commute with taking derivatives along the parameters. First they were defined and studied by M.Singer and Ph.Cassidy in the case when the field of functions in parameters is differentially closed, that is, any compatible system of differential equations has a solution in the field of functions in parameters.

We discuss a recent approach to this based on Atiyah extensions and a differential version of Tannakian categories. As an application we obtain that in a wide range of examples with a non-differentially closed field of functions in parameters, parameterized differential Galois groups and Galois correspondence still can be constructed.

Multidimensional Gauss Reduction Theory for conjugacy classes of $SL(n, \mathbb{Z})$.

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Two matrices M_1 and M_2 in $SL(n, \mathbb{Z})$ are *conjugate* if there exists a matrix X in $SL(n, \mathbb{Z})$ such that

$$M_2 = XM_1X^{-1}.$$

In our talk we study the following problem.

Problem. Describe the set of conjugacy classes in $SL(n, \mathbb{Z})$.

One of the mostly common strategies to solve this kind of problems is to find complete invariants to distinguish the classes, and further if possible to write normal form of conjugacy classes. For instance, in the similar problem for $SL(n, F)$ for an algebraically closed field F one have Jordan Normal Forms as a complete description of conjugacy classes. Jordan blocks form a complete invariant in this case. If the field is not algebraically closed, the description is much more complicated via Jordan-Chevalley decomposition.

A complete description of the set of conjugacy classes in $SL(2, \mathbb{Z})$ is given by Gauss Reduction Theory. It turns out that it is natural to consider several normal forms for a conjugacy class but not necessarily only one. Recently we showed a geometric explanation of Gauss Reduction Theory it terms of geometric continued fractions.

We extend this approach to the multidimensional case. We propose a geometric description of conjugacy classes in terms of multidimensional continued fractions in the sense of Klein-Voronoi. In the totally-real case such multidimensional continued fractions are unions of boundaries of convex hulls of all integer point inside the cones defined by invariant hyperplanes of linear operators with given matrices. These fractions introduced by F. Klein in 1895 for the totally real case. A little later G. F. Voronoi made the first attempts to generalize the construction to the rest cases.

We consider Hessenberg matrices as a multidimensional analog of reduced matrices in Gauss Reduction Theory. Hessenberg matrices are matrices that vanish below the superdiagonal. We introduce a natural notion of *Hessenberg complexity* for Hessenberg matrices, which is a nonnegative integer function, and show that each conjugacy class of irreducible matrices has only finite number of Hessenberg matrices with minimal complexity. They are all constructed starting from the vertices of Klein-Voronoi's continued fractions.

In three-dimensional case of operators with a couple of complex conjugate eigenvectors we discover the following phenomenon: Hessenberg matrices distinguish corresponding conjugacy classes asymptotically. Notice that similar statement is no longer true for the case of operators with three real eigenvalues.

Khintchine's and Jarnik's Diophantine results and their extensions.

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In 1926 A.Khintchine published his famous paper "Über eine klasse linear Diophantine Approximationen"where he find out the main phenomena of multi-dimensional linear Diophantine approximations. This paper includes results on

- aproximations with lacunary secuences;
- existence of so-called "singular systems";
- transference principles.

General theory of multidimensional Diophantine appoximations was constructed by A.Khintchine and V.Jarnik in 1920 - 1950. It happened that many results by A.Khintcine and V.Jarnik were forgotten. I suppose to give a talk about some classical results and their modern extensions and generalizations. Particulary I consider the following topics:

- theory of singular systems, especially existence extremely singular matrices;
- the phenomena of degenerate dimension of the best approximations;
- Diophantine exponents and Diophantine inequalities;
- irregularities of distribution and indefinity principles.

In 1982 W.Schmidt formulated several important unsolved problems in Diophantine approximations. Some of them were solved recently. One of the most impressive results is a solution of so-called "BADconjecture by D.Badziahin, A.Pollington and S.Velani. I intend to speak about this wonderful result and its connection to the general theory of Diophantine approximations.

Picard groupoids and reciprocity laws on algebraic surface.

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In case of a projective algebraic curve there are famous reciprocity laws for residues of a differential form and for the tame symbol of rational functions on the curve. The last reciprocity law is also called the Weil reciprocity law. These reciprocity laws are also connected with the Gauss quadratic reciprocity law and the class field theory when the curve is defined over a finite field.

There is an intrinsic proof of the Weil reciprocity law. This proof is based on the fact that the cohomology groups of coherent sheaves on an projective algebraic curve are finite-dimensional vector spaces over the ground field. Such a proof was given by E. Arbarello, C. De Concini and V.G. Kac and followed after the proof of reciprocity law for residues of differential forms given by J. Tate.

For algebraic surfaces there are Parshin reciprocity laws for two-dimensional tame symbols, [1]. These reciprocity laws are connected with two-dimensional class field theory. We give a new proof of these reciprocity laws, which has an intrinsic nature and uses adelic structures on an algebraic surface.

For this goal we construct a 2-category of torsors over arbitrary Picard groupoid. This 2-category is a 2-Picard groupoid. For any group we define the notion of central extension of this group by Picard groupoid. After that we define the analogue of commutator map and study his properties in this central extension. The commutator map is defined for any three commuting elements of the group. When we apply these constructions to two-dimensional local fields, we will obtain the new expression for two-dimensional tame symbol, which leads to the new proof of Parshin reciprocity laws on an algebraic surface in the spirit of proof of Arbarello, De Concini and Kac in case of an algebraic curve. We note that it was important for us to use non-strictly commutative Picard groupoid of graded 1-dimensional vector spaces over a field.

If we change a ground field to an Artinian local ring in these constructions, then the commutator map which was described above will give the other maps used by A.N. Parshin for explicit construction of two-dimensional local class field theory. Using our method, we obtain for these maps the reciprocity laws, which are the part of two-dimensional global class field theory. In particularly, the reciprocity laws for residue of a 2-differential form on an algebraic surface are also obtained.

This talk is based on joint results with Xinwen Zhu, [2].

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О подходящих дробях к цепным дробям до ближайшего четного.

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В работе [1] исследовались свойства цепных дробей до ближайшего четного. Алгоритм разложения вещественного числа α в них подобен алгоритму разложения в правильную цепную дробь, но на его шаге при обращении остатка меняется знак, а вместо взятия ближайшего целого берется ближайшее четное число.

Рациональные числа, для которых сумма их целочисленных числителя p и знаменателя q в несократимом представлении p/q четна, назовем *нечетными рациональными числами*. Остальные рациональные числа назовем *четными*. Подходящие дроби к четной цепной дроби суть четные рациональные числа.

В [2] был предложен ускоренный (короткий) вариант алгоритма разложения в четную цепную дробь: ряд повторяющихся шагов заменялся композицией соответствующих дробно-линейных преобразований. С n-м шагом алгоритма теперь связывалась пара четных рациональных чисел – *главная* $q_n = \dfrac{q_{1,n}}{q_{2,n}}$ и *дополнительная* $q'_n = \dfrac{q'_{1,n}}{q'_{2,n}}$ *подходящая дроби* к короткой четной цепной дроби. Считая далее, что знаменатели подходящих дробей в несократимом представлении неотрицательны, включим подходящие дроби в одну последовательность

$$\{q'_0, q_0\,, \dots q'_n, q_n\,, q'_{n+1}, q_{n+1}\,, \dots\}. \tag{13}$$

Справедлива

Теорема. *В зависимости от того, будет подходящая дробь к правильной цепной дроби числа α четным или нечетным рациональным числом, она либо встретится в последовательности (13) для α, либо равна отношению разностей числителей и знаменателей соседних подходящих дробей этой последовательности.*

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Discrete Complex Reflection Groups.

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Let k be either the field $\mathbb{R}$ of real numbers or the field $\mathbb{C}$ of complex numbers. Let V be a finite dimensional vector space over k endowed with a positive definite inner product $\langle\ ,\ \rangle$. Let $\mathbf{E}$ be an affine space over k with the space of translations V and the metric structure determined by $\langle\ ,\ \rangle$.

Definition 1. Definition An affine isometry γ of $\mathbf{E}$ is called *reflection* if its order is finite and the fixed point set $\mathbf{E}^\gamma$ is a hyperplane (i.e., $\operatorname{codim}_k \mathbf{E}^\gamma = 1$).

Definition 2. Definition A transformation group of $\mathbf{E}$ is called a *reflection group* of $\mathbf{E}$ if it is discrete and generated by reflections.

The aim of this talk is to describe a complete classification of reflection groups Γ of $\mathbf{E}$.

If either $k = \mathbb{R}$ or $k = \mathbb{C}$ and Γ is finite, such classifications are the well-known fundamental classical results due to E. Cartan, Witt, Coxeter (for $k = \mathbb{R}$), see [1], and Shephard and Todd (for $k = \mathbb{C}$), see [3]. Our contribution is the complete classification of infinite reflections groups for $k = \mathbb{C}$.

In fact, we obtain more: as a byproduct of our approach we obtain, for $k = \mathbb{C}$ and every finite reflection group W of V (i.e., a group of Shephard and Todd), the complete classification of W-invariant lattices T in V. These lattices are quite remarkable. In particular, the complex torus V/T is an abelian variety.

Our approach is also applicable for classifying reflection groups of affine spaces over quaternions and classifying lattices invariant with respect to finite quaternionic reflection groups. If time permits, I shall comment on this as well.

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Alternative algebras with hyperbolic unit loops [1]

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We investigate the structure of an alternative finite dimensional $\mathbb{Q}$-algebra $\mathfrak{A}$ subject to the condition that for some $\mathbb{Z}$-order $\Gamma \subset \mathfrak{A}$, the loop of units of $\mathcal{U}(\Gamma)$ does not contain a free abelian subgroup of rank two. We also classify the RA-loops L for which $\mathbb{Z}L$ has this property. The classification for group rings is still an open problem. This definition is an extension of the notion of hyperbolic group defined by Gromov [2] via the Flat Plane Theorem [1, Corollary $III.\Gamma.3.10.(2)$]. We also prove that if an alternative finite dimensional $\mathbb{Q}$-algebra has the hyperbolic property, then the radical of the algebra lies in its associator.

Group rings $\mathbb{Z}G$ whose unit groups $\mathcal{U}(\mathbb{Z}G)$ are hyperbolic were characterized in [5] in case G is polycyclic-by-finite. A similar question was considered for RG, R being the ring of algebraic integers of $\mathbb{K} = \mathbb{Q}(\sqrt{-d})$ and G a finite group (see [6]). In [3, 4], these results were extended to associative algebras $\mathcal{A}$ of finite dimension over the rational numbers containing a $\mathbb{Z}$-order $\Gamma \subset \mathcal{A}$ whose unit group $\mathcal{U}(\Gamma)$ is hyperbolic. An algebra $\mathcal{A}$ with this property is said to have the hyperbolic property. Using these general results, the finite semigroups S and the field $\mathbb{K} = \mathbb{Q}(\sqrt{-d})$ such that $\mathbb{K}S$ has the hyperbolic property were classified.

In this talk, our approach is the same problem in the context of non-associative $\mathbb{Q}$-algebras, in special those which are loop algebras.

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On a compactification of moduli of vector bundles by trees of bubblings of the surface: arbitrary rank case.

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We construct a non-classical algebro-geometric compactification of the scheme of moduli of Gieseker – stable vector bundles with fixed Hilbert polynomial on a smooth projective algebraic surface (S, L) over the field $k = \bar{k}$ of zero characteristic. We consider the case of arbitrary rank.

Families of locally free sheaves on the surface S are completed by locally free sheaves of some special type, on schemes which are certain modifications of S. This may be done by taking the modifications to be smooth irreducible surfaces obtained by so-called *trees of bubblings* of the surface S.

Gauge-theoretical approach to this project was provided for $k = \mathbb{C}$ by Nicholas Buchdahl [1, 2]. In his version the bubbling of the compact complex surface S at its point x means forming a (real) topological connected sum with projective plane $S\sharp\overline{\mathbb{P}}^2$ equipped with a suitable metric. The attachment is done so as the neck of the connected sum circles the point x. Bubblings can be iterated. The process of consequent bubblings is described by consequent choice of points x and then can be displayed by the union of graphs of tree type.

We provide the algebro-geometric approach to what was done by N. Buchdahl. The role of bubbling is played by blowing up of reduced point on the surface S and the role of metric in the construction is played by ample divisor class. We prove that any stable rank r coherent sheaf E can be transformed in the certain procedure into the locally free sheaf $\widetilde{E}$ on the another surface $\widetilde{S}$. This surface is obtained from S by the tree of bubblings which depends on the initial sheaf E. It is clear that this tree of bubblings is defined not uniquely. We describe the class of vector bundles to appear in the construction and propose moduli functor for pairs $((\widetilde{S}, \widetilde{L}), \widetilde{E})$. Such pair consists of bubble-tree-blown up surface $\widetilde{S}$ with distinguished ample line bundle $\widetilde{L}$ and of locally free sheaf $\widetilde{E}$ of the class described. Coarse moduli space for this functor is a projective algebraic scheme. It is birational to Gieseker – Maruyama scheme.

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О распределении приведенных базисов в двумерных целочисленных решетках.

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Приведенные базисы решеток используются при реализации и анализе различных вычислительных алгоритмов (быстрое умножение точек на эллиптических кривых, предсказание поведения псевдослучайных последовательностей и т. д.). При этом сложность и время работы алгоритма зависят от свойств приведенного базиса.

В двумерном случае распределение векторов приведенных (в различных нормах) базисов можно описать явно. В частности, можно найти плотность распределения кратчайших векторов в двумерных целочисленных решетках и плотность распределения длины второго базисного вектора.

Результат основан на применении геометрической теории цепных дробей и оценках сумм Клостермана. Он тесно связан с поведением чисел Фробениуса от трех аргументов. Для которых недавно были доказаны гипотезы Дэйвисона и Арнольда, которые также будут затронуты в докладе.

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Section “Applications”

Extended Strings: comparison of topological defects and solitons.

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Presently there exist growing interest to extended strings as simplest examples of (D+1)-dimensional nonlinear field solutions which could play crucial role in modern superstring theories as fundamental degrees of freedom. Well-known extended solutions with nontrivial topology belongs to the wide class of so-called topological defects (TD’s). Topological indices of TDЎs can be calculated from their field distributions at D-dimensional outer boundary; the examples of this kind are: 2D Abrikosov-Nielsen-Olesen (ANO) strings-vortices and 3D Polyakov-’t-Hooft hedgehogs-monopoles. An alternative possibility is to consider topological solitons (TS’s), which are found for uniform boundary conditions (contrary to TD’s) at outer (space) boundary. Their topological indices are defined by the whole solution; the well-known example of such topological solitons are 2D Belavin-Polyakov solutions and 3D Skyrme hedgehogs.

The original part of the talk is planned to be the presentation of TS-analog of the ANO TD’s, namely soliton strings-vortices in Ythe A3M modelҰ (the gauge-invariant nonlinear sigma model of the Heisenberg antiferromagnet with the "easy-axis"anisotropy, in which 3-component scalar unit isovector field interacts minimally with the Maxwell field). This A3M model possesses both Z(2) global symmetry and U(1) local symmetry which could be underlying reasons of the surprising and physically appealing properties of stable 2D A3M topological solitons, which are planned to be compared in the presentation - both with Belavin-Polyakov solutions, on one hand, and with the ANO topological defects, on the other hand.

Finally search for classically stable 3D topological solitons in realistic Quantum Field Theories (QFTs) will be shortly discussed, in particular for 3D topological solitons in bosonic sector of Weinberg-Salam theory of electroweak interactions.

Power geometry as new mathematics.

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Traditional differential calculus is effective for linear and quasilinear problems. It is less effective for essentially nonlinear problems. A linear problem is the first approximation to a quasilinear problem. The linear problem is usually solved by methods of Functional Analysis, then the solution to the quasilinear problem is found as a perturbation of the solution to the linear problem. For an essentially nonlinear problem, we need to isolate its first approximations, to find their solutions, and to construct perturbations of these solutions. This is what Power Geometry (PG) is aimed at. For equations and systems of equations (algebraic, ordinary differential, and partial differential), PG allows to compute asymptotic forms of solutions as well as asymptotic and local expansions of solutions at infinity or at any singularity of the equation (including boundary layers and singular perturbations) [1].

Algorithms of PG: (i) isolation of first approximations of the equations (via its polyhedron); (ii) simplification of the first approximations by power and logarithmic transformations; (iii) solution of the simplified equations; (iv) computation of expansions of solutions via successive linear first approximations [2].

Applications of PG: 1. Expansions of solutions to ODEs. 2. The same for Painleve equations. 3. Periodic solutions to the Beletsky equation (oscullations of a satellite). 4. Motion of the rigid body. 5. The boundary layer on a needle. 6. The restricted 3-body problem. 7. Integrability. 8. Evolution of turbulent flow.

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Delone-Hopf Triangulations in a 3-Sphere.

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Let $X \subset \mathbb{S}^3 \subset \mathbb{E}^4$ be a finite point set on a standard 3-sphere and $Del(X)$ a corresponding Delone triangulation $Del(X)$ of the sphere $\mathbb{S}^3$. It is well-known that in this case $Del(X)$ is isomorphic to the boundary $\partial(conv(X))$ of the convex hull $conv(X)$. Namely, $Del(X)$ coincides with the central projection of $\partial(conv(X))$ onto $\mathbb{S}^3$.

We call a Delone triangulation $Del(X)$ a *Delone-Hopf triangulation* if the point set X belongs to the Hopf-Clifford torus $T \subset \mathbb{S}^3$, i.e. to the torus $T := \{(\cos\varphi, \sin\varphi, \cos\psi, \sin\psi), |\ 0 \leqslant \varphi,\ \psi < 2\pi\}$. We study Delone-Hopf triangulations for sets $X \subset T$ of two sorts.

We present two main results. One is a very nice explicit description of a Delone-Hopf triangulation $Del(X)$ when a point set X is a "periodic"set. The description is given in terms of the Klein polygon for lattices of rank 2. It uses geometry of continued fractions and based on uneasy calculations and non-trivial geometric arguments.

Another main result is computing a Delone-Hopf triangulation $Del(X)$ for a random set $X \in T$, i.e. when X is a point Poisson process on the Hopf torus. The computer simulation of $Del(X)$ in this case gives a surprising phenomenon: the mean valency of vertices in $Del(X)$ grows logarithmically as the cardinality $N := |X|$ tends to infinity.

This empirical result looks even more surprisingly because it contrasts to that in a Delone triangulation for X, where X is a random set on the whole 3-sphere, the valency of a "typical"vertex $x \in X$ tends to the Meijering constant $48\pi^2/35 + 2 (= 15.53...)$. Thus, in a random 4-polytope provided all its N vertices are randomly located on the Hopf torus there are $O(N \log N)$ edges, in contrast to $O(N)$ edges in $Del(X)$ if X is a point-Poisson process on the sphere $\mathbb{S}^3$.

The Combinatorial, Contraction, and Affine Types of Parallelohedra.

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The present state of the theory on Fedorov's parallelohedra is presented. New results are given on the enumeration of combinatorial, and affine types of primitive parallelohedra.

Let Λ^d be a *translation lattice* in Euclidean space E^d with Gram matrix Q. We denote the *Dirichlet parallelohedron* by

$$\mathsf{P}(Q) := \{\mathbf{x} \in E^d \mid \mathbf{x}^t Q \mathbf{x} \leqslant (\mathbf{x}-\mathbf{t})^t Q (\mathbf{x}-\mathbf{t}), \forall \mathbf{t} \in \Lambda^d\}.$$

It is a special kind of a Fedorov parallelohedron.

The k-faces of a polytope P, are partially ordered with respect to inclusion. The k-faces of P, together with the empty set $\{\emptyset\}$, determine the *face lattice* $\mathcal{L}(\mathsf{P})$.

Definition 1. *Two polytopes* P *and* P' *are combinatorially equivalent,* $\mathsf{P}' \overset{comb}{\simeq} \mathsf{P}$*, and belong to the same combinatorial type, if there exists a combinatorial isomorphism* $\tau : \mathcal{L}(\mathsf{P}) \to \mathcal{L}(\mathsf{P}')$.

A *zone* of P is a set of parallel 1-faces of P. A zone Z is called *closed* if every 2-face of P contains either two edges of Z, or else none. Otherwise Z is called *open.* The edges of a zone Z are collected into subsets $\mathsf{S}_j^{\mathsf{Z}}$, $j = 1, ..., s(\mathsf{Z})$ according to their length l_j,$l_1 < l_2 < \cdots < l_{s(\mathsf{Z})}$. Each subset contains a multiple of $2d$ edges. By *zone contraction* $\mathsf{P}^{\downarrow}$, we understand the process of contracting every edge of a closed zone Z by the amount of its shortest edges $\mathsf{S}_1^{\mathsf{Z}}$ [3]. A parallelohedron P_c is said to be *totally contracted* if all its zones are open. A parallelohedron P_m is *maximal* if it cannot be obtained by a zone contraction of any other parallelohedron. The *zone extension* $\mathsf{P}^{\uparrow}$ is the inverse operation of zone contraction.

Each maximal parallelohedron defines a complete *zone-contraction lattice* $\mathcal{Z}(\mathsf{P}_m)$ by contracting all combinations of closed zones. Each relatively, or totally zone-contracted parallelohedron P_c defines a *zone-contraction family* $\widehat{\mathcal{Z}}(\mathsf{P}_c)$.

Definition 2. *Two parallelohedra* P *and* P' *are contraction equivalent,*$\mathsf{P}' \overset{contr}{\simeq} \mathsf{P}$*, and belong to the samecontraction type, if*
i) there exists a face lattice isomorphism $\kappa : \mathcal{Z}(\mathsf{P}) \to \mathcal{Z}(\mathsf{P}')$*;*
ii) there exists a combinatorial isomorphism for each $\widetilde{\mathsf{P}} \in \mathcal{Z}(\mathsf{P})$, $\kappa : \widetilde{\mathsf{P}} \mapsto \widetilde{\mathsf{P}}' \overset{comb}{\simeq} \widetilde{\mathsf{P}}$.

In dimensions $d < 5$, both classifications coincide.

Still a finer classification of parallelohedra is given by affine equivalence. Let $\mathsf{A} = \{a_{ij}\}$ be a non-singular $d \times d$ matrix with real coefficints a_{ij}.

Definition 3. *Two polytopes* P *and* P' *are affinely equivalent,* $\mathsf{P}' \overset{aff}{\simeq} \mathsf{P}$*, and belong to the same affine type, if there exists an affine mapping* $\mathsf{A} : \mathsf{P}' = \mathsf{A}\mathsf{P}$.

Vorono¨**i** [7] conjectured that every parallelohedron is affinely equivalent to a Dirichlet parallelohedron, and he proved it for primitive parallelohedra.

We shall partition the open cone of positive definit quadratic forms $\mathcal{C}^+$, into connected open *subcones of equivalent types of parallelohedra*

$$\Phi^+(\mathsf{P}) = \{Q \in \mathcal{C}^+ \mid \mathsf{P}(Q) \simeq \mathsf{P}\}.$$

For the determination of the equivalence of two parallelohedra we note that all $\mathsf{P}(Q)$ within a subcone of combinatorial type have identical face lattices. This allow us to construct relative equivalence schemes. For combinatorial equivalence, the boundary of a subcone is give by the condition that at least $d+1$ facets meet in the common vertex $\mathbf{v} \subset \mathsf{P}$. By writing down for each facet the numbers of all subordinated vertices in increasing order, we obtain the *relative polytope scheme* to characterize the combinatorial type. For contraction, and affine equivalence, additional boundaries of the corresponding subcone are given by the condition that for a zone Z at least two subsets $\mathsf{S}_h^{\mathsf{Z}}$, and $\mathsf{S}_k^{\mathsf{Z}}$ have equal length, $l_h = l_k$. By comparing for each zone Z_i the lengths of each subset $\mathsf{S}_h^{\mathsf{Z}_i}$ with each other $\mathsf{S}_k^{\mathsf{Z}_i}$ we obtain a sequence of comparison operators $<, =, >$, which determines the *relative affine scheme*. It holds that $\Phi_{aff} \subseteq \Phi_{contr} \subseteq \Phi_{comb}$

We designed algorithms to determine the various kinds of subcones. To sum up the results of Fedorov [5], Delone [1], Shtogrin [6], and ours [2], [3], [4], we obtain:

Theorem. *In E^2, there exist 2 combinatorial types of parallelogons, and in E^3, there exist 5 combinatorial types of parallelohedra. In E^4, there exist 52 combinatorial types of paralllelohedra which belong to 2 zone-contraction families. In E^5, there exist 179'372 contraction types of parallelohedra which belong to 82 zone-contraction families. They belong to 103'769 combinatorial types.*

Previous results in E^6 show that there exist much more than 198'000'000 combinatorial types of primitive parallelohedra. We started to calculate subcones of affine types which, in some cases, prove to have a very complicated combinatorial structure.

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3D reconstructions of synaptic structures of central neurous system using Delaunay partitionings[1].

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Reliable 3d reconstruction of synaptic structures of CNS using high resolution electron microscopy data is hard unsolved problem of modern biophysics and numerical geometry. State of the art technologies allow to obtain sections of the brain tissue with thickness up to 10 nanometers, while spatial resolution for each slice (pixel size) can be as small as 2 nm. This resolution allows to view directly synapses or so-called postsynaptic densities (PSD) which serve as a basic recognition patterns of a contact between axon and dendrite. Most PSD are located at special dendritic structures called dendritic spines. Manual detection and marking of PSD and dendritic spines on EM images is very time consuming and requires high level of expertise in neurobiology. Contours on the images still have to be detected manually. Fig. 3 shows such sample image. Analysis, classification, geometrical and topological description of dendritic spines and PSDs is recognized as a powerful tool for investigation of the mechanisms of memory [1].

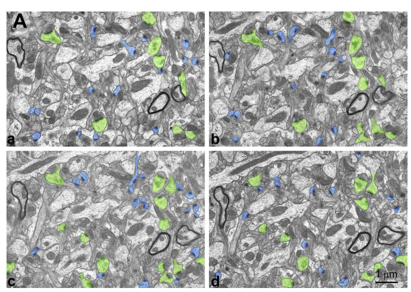

Fig. 3. Electron microscopy image of brain tissue sections, color marks dendritic spines.

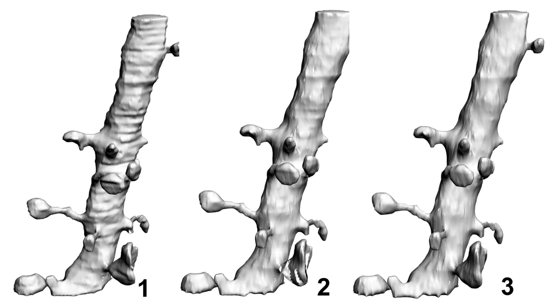

Fig. 4. Reconstruction of dendrite from cross sections with simultaneous aligment of contours, multiple dendritic spines are clearly visible.

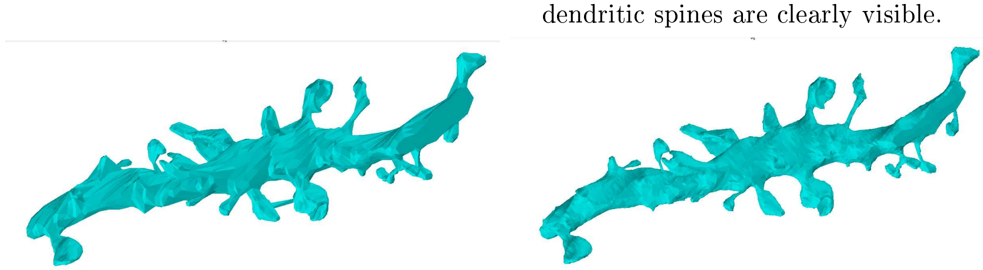

Fig. 5. Direct comparison of reconstruction results: left - Trace algorithm, right - suggested algorithm.

In order to reconstruct surfaces from contours on two consecutive cross sections we use constrained planar Delaunay triangulation for a set of flat contours. Resulting triangulation is mapped into 3d space forming membrane-like surface spanning spatial contours [2]. Nice feature of this algorithm is that it allows to avoid topological errors when reconstructing branched

[1] This is a joint work with A.I. Fedotov (Moscow Institute of Physics and Technology), L.N. Kudryavtseva (Moscow Institute of Physics and Technology), I. Patrushev (Institute of Cell Biophysics RAS, Puschino), R.V. Polozov (Institute of Theoretical and Experimental Biophysics RAS, Puschino), V.I. Popov (Institute of Cell Biophysics RAS, Puschino).

configurations. Consecutive EM images can contain alignment uncertainty. In order to resolve it local alignment for all pairs of sections is applied which generally result in jagged surfaces shown in fig.4, left. To resolve this problem we construct transformations of section contours via discrete curvature functionals minimization for selected objects spanning geometrical scene. Such "smoothing" is illustrated in fig.4, center-right. As a basis set of objects one generally chooses a subset of mitohondria and axons.

Currently neurobiologists use algorithm "Trace" [3] for image alignment and recostruction. It is well known that this algorithm is prone to geometrical and topological errors. Comparison with suggested algorithm is illustrated on Fig. 5

Fig. 6 illustrates complexity of the problem. Preparation of data for geometrical scene containing fragment of neural network with several dendrites, axons and astrocites and hundreds of PSDs took several months.

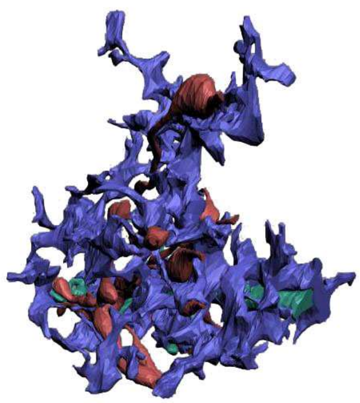

Fig. 6. Fragment of reconstructed brain tissue: dendrites, axons and astrocites.

Next step in our research should be computation of geodesic distance matrices between different PSD providing rough estimates for signal traveling times as well as construction of 3d computational meshes for such scenes which is the prerequisite for numerical simulation of signal transmission, spill-over and diffusion of neuromediators. This meshing problem is quite hard and requires new versions of existing meshing algorithms.

In order to attain reliable reconstruction of the surface features and estimation of discrete curvatures we develop duality-based method which reconstructs simultaneously a pair of locally polar polyhedra [4]. Fragments are shown in fig. 7. One should note that construction of dual polyhedra is quite hard problem which implies simultaneous optimization of primal polyhedron and its iterative decomposition into convex and saddle subdomains.

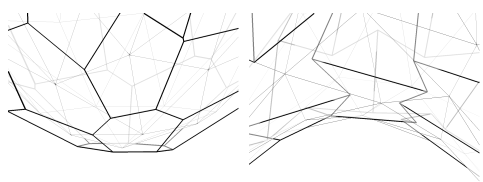

Fig. 7. Dual polyhedral approximants for convex and saddle surfaces.

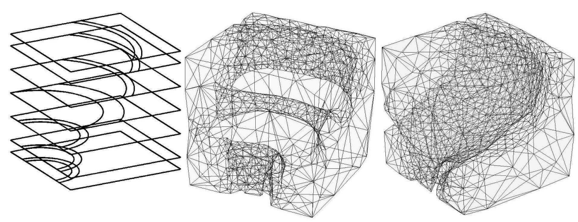

Fig. 8. Nonunique solution of reconstruction problem.

In order to construct 3d meshes we have developed special variant of Delaunay partitioning technique in implicit domains defined by non-smooth implicit function. This technique allows for automatic sharpening of boundary edges without their explicit detection. It allows to construct 3d meshes directly from the set of cross sections and in general from combination of analytical definition with a "soup" of points, segments and faces. Fig. 8 illustrates how this technique does the task of simultaneous meshing and reconstruction. The set of sections assumes the presence of oblique cut, however both implicit function reconstruction and 2.5d Delaunay triangulation based reconstruction result in multiple bridges/tunnels. In order to resolve this problem we have developed a preliminary version of reconstructor based on special control vector fields partially aligned with oblique cuts, see fig. 8, right. However formalization of biological requirements for reconstruction still provides considerable difficulties. This work is partially supported by RFBR grant ofi_m 09-01-12106.

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О геометрии системы точек Делоне-Александрова.

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Интересующая нас система точек Делоне-Александрова суть система центров равных шаров, образующих упаковку. Поэтому задача Кеплера, нахождения плотнейшей упаковки равных шаров, имеет аналогичную формулировку для системы точек Делоне-Александрова. Интререс к проблеме Кеплера укрепился после включения ее в число важнейших проблем Д.Гильбертом. Решение проблемы Кеплера, даже после появившихся публикаций, претендующих на исчерпывающее решение проблемы, остается трудной задачей, так как изучение плотных упаковок не дало нового вывода о плотных упаковках, кроме вывода Кеплера. Плотнейшая упаковка Кеплера, известная сейчас как гранецентрированная плотнейшая упаковка равных шаров, однозначно характеризуется своей единственной областью Вороного и двумя полиэдрами Делоне. Мы предлагаем обобщенный алгоритм Коксетера нарушения порядка расположения точек по способу гранецентрированного кубического кристалла. Данный алгоритм генерирует гомологический ряд почти периодических(почти кристаллографических) способов расположения точек Делоне-Александрова в трехмерном евклидовом пространстве. При малом параметре алгоритма Коксетера найден полный списот пустот Делоне и областей Вороного модели.

The estimation of gradient approximation on Delaunay triangulation.

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1. Desingate the convex hull of $k+1$ points $p_i, i = 0, ..., k \leqslant n$ such as vectors $p_1 - p_0, p_2 - p_0, ..., p_k - p_0$ are linearly independent as k-dimensional simplex S in $\mathbf{R}^n$.

Suppose $D \subset \mathbf{R}^n, n > 1$ is a domain, which has a defined sequence $\{P_m\}$ of finite sets of points. Let's examine the triangulation T_m for each of these sets. We mean the set $\{S\}$ of n-dimensional simplexes S as the triangulation of set of points, such as:

1) Each point $p_i \in P_m$ of the defined set is a vertex of one of the simplexes S;
2) Each vertex of any simplex S is one of the points $\{p_i\} \in P_m$;
3) Interiority of intersection of any two simplexes is empty;
4) There is the only one simplex S, which satisfies the conditions 1) – 3).

The triangulation of points set is the *Delaunay triangulation* (see [1], [2]), if the circumsphere of each simplex of triangulation contains none of points of this set.

Let's call the triangulation as *acute*, if all of its angles between any pair of its adjacent k-dimensional faces are acute for each simplex.

It's easy to verify, that any acute triangulation is the Delaunay triangulation.

For each simplex $S \in T_m$ let's define the length of its maximum side d_S. Let's assume

$$d_m = \max_{S \in T_m} d_S.$$

We'll examine such sets of points P_m and their triangulations T_m, which satisfy the following conditions:

$$d_m \to 0 \text{ when } m \to \infty. \tag{14}$$

$$\forall \varepsilon > 0\ \exists m_0 \in \mathbf{N} : \forall m > m_0 \text{ and } \forall x \in D\ \exists a \in P_m \text{ such that } |a - x| < \varepsilon. \tag{15}$$

The second condition means that P_m is an ε-net for all sufficiently great m. Let's examine a function $f(x), x \in D$ belonging to the class $C^1(D)$. For the defined triangulation T in the domain D let's build up a piecewise affine function $f_T(x)$ such that

$$f_T(a) = f(a), \text{ for any vertex } a \text{ of triangulation } T.$$

It's easy to prove, that when the conditions (14) and (15) are satisfied, the sequence $f_T(x)$ converge uniformly to function $f(x)$ on every compact subset $U \subset D$. This paper studies the possibilities of the Delaunay triangulation for the approximation of the gradient of the function $f(x)$ by the gradient of $f_T(x)$ and also investigates admissible generalizations.

2. The following theorem gives a quantitative characteristic of the approximation property of the Delaunay triangulation. It is necessary to note, that the properties (14) and (15) of triangulations T_m are not enough to get such estimates. This fact is demonstrated by the classical Schwartz's example (see [3]), where the square of the side surface of quadric cylinder is calculated. We proved the following result:

Theorem 1. *Suppose the Delaunay triangulation T of some ε-net of plain domain $D \subset \mathbf{R}^2$ is defined, which satisfies the condition (15). Then for any compactly embedded subset $U \subset\subset D$ the following estimate is correct:*

$$\max_{S \in T, S \subset U} \max_{x \in S} |\nabla f(x) - \nabla f_T(x)| \leqslant \max_U \max_{1 \leqslant i,j \leqslant 2} \left| \frac{\partial^2 f(x)}{\partial x_i \partial x_j} \right| (4 + 5\sqrt{2})\varepsilon.$$

In case $n = 3$ an analogical result can be obtained only for much stricter triangulation class than the Delaunay triangulation. There is the

Theorem 2. *Suppose T is an acute triangulation of some ε-net in the domain $D \subset \mathbf{R}^3$, which satisfies the condition (15). Then for a function $f(x) \in C^2(D), x \in D$ and for a compactly embedded subset $U \subset\subset D$ the following estimate is correct:*

$$\sup_{S \in T, S \subset U} \sup_{x \in S} |\nabla f(x) - \nabla f_T(x)| \leqslant \max_U \max_{1 \leqslant i,j \leqslant 3} \left| \frac{\partial^2 f(x)}{\partial x_i \partial x_j} \right| (3 + 9\sqrt{3})\varepsilon.$$

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Adiabatic limits and problems of distribution of integer points[1]

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1. Adiabatic limits. Let $(M, \mathcal{F})$ be a closed foliated manifold equipped with a Riemannian metric g. So we have a decomposition of the tangent bundle TM of M into a direct sum $TM = T\mathcal{F} \oplus T\mathcal{F}^{\perp}$, where $T\mathcal{F}$ is the tangent bundle of $\mathcal{F}$ and $T\mathcal{F}^{\perp}$ is the orthogonal complement of $T\mathcal{F}$. Accordingly, the metric g can be written as a $g = g_F + g_H$ of the tangential component g_F and the transversal component g_H. Define a one-parameter family g_ε of Riemannian metrics on M by the formula

$$g_\varepsilon = g_F + \varepsilon^{-2} g_H, \quad \varepsilon > 0.$$

For any $\varepsilon > 0$, consider the Laplace-Beltrami operator Δ_ε on M determined by g_ε. Its spectrum is a countable set of eigenvalues with finite multiplicity $0 \leqslant \lambda_0(\varepsilon) \leqslant \lambda_1(\varepsilon) \leqslant \ldots$ such that $\lambda_j(\varepsilon) \to +\infty$ when $j \to \infty$. Let us define the eigenvalue distribution function of Δ_ε by

$$N_\varepsilon(\lambda) = \sharp\{i : \lambda_i(\varepsilon) \leqslant \lambda\}, \quad \lambda \in \mathbb{R}.$$

In the case when $\mathcal{F}$ is a Riemannian foliation and g is a bundle-like metric, we proved an asymptotic formula for $N_\varepsilon(\lambda)$ when $\varepsilon \to 0$ (see [1]).

Following Witten, the limit $\varepsilon \to 0$ is called adiabatic limit.

2. Distribution of integer points. Let F be a p-dimensional linear subspace of $\mathbb{R}^n$ and $H = F^{\perp}$ the orthogonal complement of F with respect to the standard Euclidean metric in $\mathbb{R}^n$. For any $\varepsilon > 0$, we define a linear transformation $T_\varepsilon : \mathbb{R}^n \to \mathbb{R}^n$ by the formula

$$T_\varepsilon(x) = \begin{cases} x, & \text{if } x \in F, \\ \varepsilon^{-1} x, & \text{if } x \in H. \end{cases}$$

For any bounded domain S in $\mathbb{R}^n$, denote

$$n_\varepsilon(S) = \#(T_\varepsilon(S) \cap \mathbb{Z}^n), \quad \varepsilon > 0.$$

We are interested in the asymptotic behavior of $n_\varepsilon(S)$ when $\varepsilon \to 0$.

3. Relation with adiabatic limits. The problems mentioned above are related as follows. As above, let F be a p-dimensional linear subspace of $\mathbb{R}^n$. Consider the n-dimensional torus $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$ and the linear foliation $\mathcal{F}$ on $\mathbb{T}^n$ determined by F: the leaf L_x of $\mathcal{F}$ through $x \in \mathbb{T}^n$ has the form

$$L_x = x + F \mod \mathbb{Z}^n.$$

Let g be the standard flat metric on $\mathbb{T}^n$, and let g_ε be the family of Riemannian metrics on $\mathbb{T}^n$, which determines the adiabatic limit.

Denote by $B_r(0)$ the ball in $\mathbb{R}^n$ of radius r centered at the origin. For any $\lambda > 0$, the number $n_\varepsilon(B_{\sqrt{\lambda}}(0))$ of integer points in the ellipsoid $T_\varepsilon(B_{\sqrt{\lambda}}(0))$ is related with the eigenvalue distribution function $N_\varepsilon(\lambda)$ of the Laplace-Beltrami operator Δ_ε associated with g_ε by the formula

$$n_\varepsilon(B_{\sqrt{\lambda}}(0)) = N_\varepsilon(4\pi^2\lambda).$$

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In particular, the general results on adiabatic limits for Riemannian foliations mentioned above imply an asymptotic formula for $n_\varepsilon(S)$ as $\varepsilon \to 0$ in the case when $S = B_{\sqrt{\lambda}}(0)$.

4. The results. First, we prove an asymptotic formula for $n_\varepsilon(S)$ when $\varepsilon \to 0$ in the case when S is an arbitrary bounded domain in $\mathbb{R}^n$ with smooth boundary. Next, under some additional assumptions on S, we state a refined remainder estimate in the asymptotic formula for $n_\varepsilon(S)$. Finally, using these results, we obtain more precise remainder estimates in the asymptotic formula for the eigenvalue distribution function $N_\varepsilon(\lambda)$ of the Laplace-Beltrami operator Δ_ε in adiabatic limit for the particular case of the linear foliation on the torus.

This is joint work with A.A. Yakovlev (see [2]).

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Многослойная модель в оптике. Новые аналитические результаты.

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Одной из классических задач оптики является анализ распространения света в слоистой среде [1]. Мы рассмотрим плоский волновод, представляющий собой совокупность $m > 2$ слоцв диэлектриков с показателями преломления n_j, $1 \leqslant j \leqslant m$. Пусть в крайних слоях бесконечной толщины $n_1 \geqslant n_m$. А среди внутренних слоцв имеется слой с показателем преломления большим n_1. Как известно [1, 2], существует два типа электромагнитных волн в волноводе: TE- и TM-волны. Если считать слои перпендикулярными оси Ox, а волны распространяющимися вдоль оси Oz, то в j-ом слое уравнение для — амплитуды составляющей $E_y(x)$ вектора электрической напряжцнности гармонической TE-волны записывается [2] так

$$\frac{c^2}{\omega^2}\frac{d^2E_y}{dx^2} + n_j^2E_y = \beta^2E_y,$$

где β — эффективный показатель преломления волновода (постоянная распространения волны), ω — частота волны, c — скорость света в вакууме. В каждом слое решение этого уравнения есть либо линейная комбинация синусоиды и косинусоиды, либо линейная комбинация двух экспонент в зависимости от знака величины $\beta^2 - n_j^2$. На граничных плоскостях слоцв решения сшиваются по условиям непрерывности величин E_y и $\frac{dE_y}{dx}$. Если при некотором значении β получается решение, у которого в крайних бесконечных слоях поле экспоненциально убывает на бесконечности, то это значение постоянной распространения β называется собственным, а соответствующее решение — собственной TE-модой волновода. Нахождение собственных мод является одной из основных задач теории волноводов.

Известные уравнения для нахождения собственных значений β (их называют дисперсионными) с трудом поддаются анализу и решению при числе слоцв волновода большем четырцх. Поэтому были развиты весьма эффективные численные методы решения задачи. Однако, численные методы не могут дать результатов качественного характера. Автором была предложена новая форма дисперсионного уравнения, которую он назвал многослойным уравнением [3, 4]. Многослойное уравнение обладает рядом преимуществ перед известными дисперсионными уравнениями. С помощью несложной программы, реализованной на персональном компьютере, его можно индуктивно выписывать и решать для волноводов, содержащих, во всяком случае, до двух десятков слоцв.

В отличие от известных дисперсионных уравнений многослойное уравнение довольно легко поддацтся математическому анализу. Путцм его анализа автором были получены [5] точные и эффективные формулы для числа собственных электромагнитных TE- и TM-мод в произвольных плоских диэлектрических волноводах. Причцм, эта задача свелась к геометрической задаче подсчцта поворота годографа некоторой вектор-функции вокруг начала координат в плоскости. Подсчцт числа мод для волноводов, содержащих десятки слоцв, с помощью этих формул на персональном компьютере происходит практически мгновенно.

Из полученных формул вытекает следующая теорема. Пусть показатели преломления слоцв чередуясь принимают два значения: $n_1 < n_2$, а толщины внутренних слоцв чередуясь принимают значения u и v.

ТЕОРЕМА. Каковы бы ни были величины $u, v, n_1 < n_2$ *, при неограниченном росте числа слоев волновода число собственных TE и TM-мод в нем также неограниченно растет.*

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Tube method — An integral-geometric approach to statistical distribution theory[1].

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Let $X(p)$, $p \in M$, be a real-valued Gaussian random field defined on a finitely dimensional index set M with mean 0 and variance 1. Consider approximating upper tail probability of the maximum of $X(p)$ over M,

$$P(c) = \Pr\left(\sup_{p\in M} X(p) \geq c\right). \tag{1}$$

Here we restrict our attention to a class of random fields of the form

$$X(p) = \langle p, \xi \rangle, \quad p \in M \subset \mathbb{S}^{d-1}, \tag{2}$$

where $\xi = (\xi_1, \ldots, \xi_d)$ is a random vector consisting of independent standard Gaussian random variables, and $\mathbb{S}^{d-1}$ is the set of unit vectors in $\mathbb{R}^d$. This is a canonical presentation of Gaussian fields with finite dimensional covariance functions. Although this is a restricted class of Gaussian fields, this covers a wide range of distributions appearing in multivariate analysis.

The set of points in $\mathbb{S}^{d-1}$ whose geodesic distance from M is less than or equal to θ is called the tube around M with radius θ, and is denoted by $\mathrm{Tube}(M, \theta)$. Write $d-1$ dimensional volume of the tube as

$$V(\theta) = \mathrm{Vol}_{d-1}(\mathrm{Tube}(M, \theta)).$$

The relation between $P(\cdot)$ and $V(\cdot)$ is essentially the Laplace transform and its inversion. This implies that the asymptotic behaviors of $P(a)$ as $a \uparrow \infty$ and $V(\theta)$ as $\theta \downarrow 0$ are determined by each other. The basic strategy of the tube method is to evaluate $V(\theta)$ for θ small first, and then to obtain $P(c)$ for c large by the Laplace transform.

If M is a piecewise-smooth submanifold of $\mathbb{S}^{d-1}$, then the volume $V(\theta)$ can be evaluated by means of integral-geometric approaches. In particular, when M is a closed Riemannan manifold, the volume formula is described in terms of Weyl's geometric quantities or Lipschitz-Killing curvatures (Weyl's tube formula) ([5]).

For the purpose of approximating the probability (1), another method referred to as Euler characteristics heuristic (EC heuristic) is known. In this method, the expectation of Euler characteristic of a excursion set, $E\left[\chi\{p \in M \mid X(p) \geq c\}\right]$, is used as an approximation to (1). For the class of Gaussian fields (2), it can be proved that the EC heuristic is essentially the same as the tube method by extending Morse's theory ([1], [6]).

Let $K \subset \mathbb{R}^d$ be the cone with the base set M. Then, $M = K \cap \mathbb{S}^{d-1}$, and $\max_p X(p) \vee 0$ is the length of orthogonal projection of ξ onto the cone K in $\mathbb{R}^d$. For various cones K that are statistically interesting, the volume formula $V(\theta)$ and the upper probability $F(c)$ can be obtained explicitly. The following are typical examples:

(i) $K = \{h_1 \otimes h_2 \mid h_i \in \mathbb{S}^{d_i - 1}\}$, where $\otimes$ is the Kronecker product. The maximum $\max_p X(p)$ is stochastically equal to the square root of the largest eigenvalue of a $d_1 \times d_1$ Wishart matrix with d_2 degrees of freedom ([4]).

(ii) $K = \{(h_1 \otimes h_2 - h_2 \otimes h_1)/\sqrt{2} \mid H = (h_1, h_2) \in V_{2,d}\}$, where $V_{2,d}$ is a Stiefel manifold. The maximum $\max_p X(p)$ is stochastically equal to the largest singular value of $d \times d$ skew symmetric Gaussian random matrix ([2]).

[1] Joint work with Akimichi Takemura (Univ. of Tokyo) and Naohiro Kato (Graduate Univ. for Advanced Science).

(iii) $K = \{h^{\otimes k} \mid h \in \mathbb{S}^{d-1}\}$. The distribution of $\max_p X(p)$ is equivalent to the limiting distribution of the sample k-th cumulant ([4]).

(iv) Let $z \sim N_d(\mu, I_d)$. Based on the observation z, consider a statistical hypothesis testing for a null hypothesis $H_0 : \mu = 0$ against $H_1 : \mu \in K$, where K is a cone in $\mathbb{R}^d$. Then, the null distribution of the likelihood ratio test is the distribution of $(\max_p X(p) \vee 0)^2$. The following cones are interesting in this context:

$$K = \{(\mu_i) \in \mathbb{R}^d \mid \mu_1 \le \cdots \le \mu_d\} \quad \text{(simple order cone)},$$
$$K = \{A \in \mathbb{R}^{d\times d} \mid A \succeq 0\} \quad \text{(cone of positive semidefinite matrices, [3])},$$
$$K = \left\{(c_i) \in \mathbb{R}^d \mid \textstyle\sum_i c_i x^{i-1} \ge 0,\ \forall x \in [a,b]\right\} \quad \text{(cone of positive polynomials)}.$$

The obtained formulas for the upper tail probability (1) are very accurate when c is moderately large, at least, and are practical enough for the purpose of calculating p-values in testing statistical hypotheses.

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About definition of singular transformation by N.V. Efimov.

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Abstract. The base of informatics-linguistic interpretation of applied geometry is produced briefly.

An author of this article goes in precision of geometrical models in CAD/CAM. Points of ellipses crossing is a need to compute for few of geometrical model of machine-building goods. Let are two ellipses $E_1 \equiv< x_1, y_1, a_1, b_1, \varphi_1 >$ and $E_2 \equiv< x_2, y_2, a_2, b_2, \varphi_2 >$, when centers of ellipse $(x_j, y_j) \in \mathbb{R}\times\mathbb{R}$, semiaxes and tilt angle $a_j, b_j, \varphi_j \in \mathbb{R}$, $j \in \{1,2\}$, $\varphi_j \in [-\pi, \pi]$. Ellipses may be representing in quadratic form $c_jx^2 + 2d_jxy + e_jy^2 + n_jx + k_jy + l_j = 0$, when $c_j, d_j, e_j, n_j, k_j, l_j \in \mathbb{R}$. The task comes to solving of equation of four powers. There equations are finding with Descartes-Euler and Cardano-Tartaglia methods one after another. In most cases the solutions are receive with complex component $x_j = r_j + \mathbf{i}s_j$, when $r_j, s_j \in \mathbb{R}$, $j \in \mathbb{Z}^+$. Sometimes a value of complex component s_j congruent with r_j. Therefore, an author wanted to find a solution without complex number.

Method of transformations chain is existed in applied geometry. This method allows breaking matrix of unrestricted linear transformation to sequence of name conversions. Chain of transformations of ellipses cross point was founded. One of transformation is shear conversion.

Classic method of finding of transformation parameters of quadratic form obtain coefficients k_x and k_y in proper basis [1]. Basis depend on proper angle of quadratic form α. The parameters is find in series: α, k_x and k_y. N.V. Efimov [1, p. 128] was quote that transformation $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ is not have a proper basis. Definition of transformations chain is not possible by classic means.

The research was moved in orthonormal basis for quadratic form with canonic equation. The representation of ellipse was selected by parametric equations set. Canonic equation of ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ have parametric equations set $\begin{cases} x = a\cos t \\ y = b\sin t \end{cases}$. Author's student was alighted an approach to finding of parameters of arbitrary linear transformation of ellipse by additional angle β [2, p. 50]. Method get parameters of transform ellipse by solution of parametric equations set: $\begin{cases} R(k_xf_x(t), \alpha) = aR(f_x(t), \beta) + hR(f_y(t), \beta) \\ R(k_yf_y(t), \alpha) = gR(f_x(t), \beta) + bR(f_y(t), \beta) \end{cases}$, when $R(f, \varphi)$ – rotation transformation.

Method is work the linear-independed transformations the singular conversions. Theoretical foundation of this is next to Efimov definition of singular transformation. He writes [1, p. 94]: singular transformation convert the plane to line. On this base was set a hypothesis of preservation of permutation symmetry $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ for any reflection on plane [2]. Next step in our research is problem what contradict to store permutation symmetry. Cartesian product is solved this issue from set theory and relational algebra. There foundations were resulted table of binary symmetry of Euclidean plane.

Developing the Leibniz statement about special space relationships (automorphisms), H. Weil proposed the table of symmetries [3]. The table is definite symmetries on plane and space. The translational symmetry extract particularly. She was not included in the table.

Considering the axiomatics of Euclidian plane, Diuedonne [4] determines two types of the symmetry: the symmetry (the permutation symmetry) and the mirror symmetry. Besides, he widely uses the identity transformation unitary matrix(the symmetry of existence). Treating the relationships, caused by the automorphisms, Diuedonne inclines to the interpretation of Bachmann-Yaglom symmetries as binary symmetries.

The notion of automorphism developed in the set theory. A. Fraenkel constructs the ZFC axiomatics on the definition of automorphic relationship of set membership of element [5]. While, the principle of nonempty set existence (the existence symmetry) is indirectly presented in this definition. The ZFC axiomatics doesn't include any axioms, which determine the order of set. Using the same objects, the Codd relational algebra, on the contrary, suggested the existence of the order set before the determination of the set. This demand is determined by the Codd second rule or by the first normal form of relational table. The relational algebra has a lot of empiricisms. In spite of it, the thought of primacy of order and secondary of the set is taken as a basic statement of new interpretation.

The table of the binary symmetries of the Euclidian plane is defined:

1. Existence from set theory (Zermelo) and geometry (Duedonne);
2. Set membership from set theory (A. Fraenkel);
3. Linguistic order from geometry (Descartes, Klein) and relational algebra (Codd);
4. Mathematical order from set theory (Cantor) and geometry (H. Weil);
5. Permutation from geometry (Gilbert, Duedonne);
6. Mirror from geometry (Gilbert, Duedonne) and art (Vitruvius, Leonardo).

First two types of automorphism cover to the set theory and influence on practically all geometric problems. Let's go into detail on the 3rd and the 4th types of the symmetry. Let's consider the set $\mathbb{Z}$. This set obeys to the symmetry of translation were the step (rythm) is equal 1. There is now step between numbers for the set $\mathbb{R}$, but, nevertheless, there is the symmetry of translation, because every number is more than previous and less than next number, as it was showed by Cantor. Let's consider the set of coordinates names in space: X, Y, $Z, \ldots$ Every name is unique, as every real number is unique. The order of sequence is accurately defined. There is no rhythm as for real numbers as for the names of coordinates. The set of names makes the symmetry of translation. For unification, let's call this symmetry the symmetry of order. Let's call the order on numbers mathematical and the order of names – linguistic. Since, names can be arbitrary (but unique), they may not form sets, but be enumerable by some means.

The order of sequence in table is strict. The chosen symmetry can not contradict the older one. Thereby, the empty set is the only asymmetry; it contradicts the symmetry of existence. Two theorems, defining the priority and interaction of the 4th and the 5th automorphisms are proved; the hypothesis of the symmetries balance is preconceived.

The hypothesis of symmetries balance.

Any relation, refection, function, operation, operator, morphism, transformation on Euclidean plane is carrying out accordingly in order to execute permutation symmetry with preservation of mathematical order symmetry.

Results of theoretical mathematics are use in table only. Additional semantics may be produce for any parts of geometry from the informatics-linguistic interpretation. The definition of canonic equation is gain a strong substantiation. Opposition between parabola and others conic sections (according P.S. Alexandrov) is dismount. Some results were obtained in linear transformations of complicated forms. Connection interpretation with theory or orthogonal invariant allows to make this.

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Periodic packings of d-dimensional polycubes[1].

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Consider some d-dimensional lattice L. A polycube is a finite union of elementary cells of L with connected interior. Centers of elementary cells from polycube are called polycube points. The packing of polycubes is called normal if all polycube points from the packing belong to L. The polycubes packing is called periodic if its automorphism group contains some d-dimensional lattice Γ. If the fundamental domain of Γ contains only single polycube we have a translation polycube packing. Let k be a packing density. If $k = 1$ we have a polycube tiling.

A packing space [1] is the pair (L, w), where L is a lattice, and w is a function $w : L \to \{0, 1, \dots n-1\}$ such that all sets $w^{-1}(i)$, $i = 0, 1, \dots, n-1$ are equivalent by translation to some sublattice $\Gamma \subseteq L$. For any lattice point $x \in L$ $w(x)$ is called the weight of this point. The number n is called the order of packing space. It is obvious that $n = [L : \Gamma]$.

Theorem 1. *Let $P = \{\beta_i\}_{1\le i\le r}$ be d-dimensional polycube. Then the following conditions are equivalent:*

1) There exists the polycube packing of P with packing density $k = \frac{r}{n}$

2) There exists the packing space (L, w) of order n and vector x_0 such that weights of the points $\{\beta_i + x_0\}_{1\le i\le r}$ are pairwise different.

Using this theorem we obtain the algorithm which generates all translation packings of a given polycube with a given packing density. Let $C_d(n)$ be a computational complexity of this algorithm.

Theorem 2. *In two-dimensional case we have*

$$C_2(n) = O(n^2 \ln \ln n),$$

$$\frac{1}{n}\sum_{i=1}^{n} C_2(i) = O(n^2).$$

In d-dimensional case we have

$$C_d(n) = O(nI_d(n)),$$

where $I_d(n)$ is a number of sublattices of $\mathbb{Z}^d$ with the index n.

Note that the exact formula for $I_d(n)$ was obtained by B.N.Delone in [2].

Now let $T_d(n)$ be a number of d-dimensional translation polycube tilings, where n is a volume of polycube.

Theorem 3. *There constants c_1, c_2 exist such that*

$$c_1 2^n \le T_2(n) \le c_2 2,7^n.$$

The theorem 1 can be generalized to finite sets of polycubes. Consider a finite sets of polycubes $\{P_j\}_{1\le j\le M}$, where $P_j = \{\beta_{ij}\}_{1\le i\le r_j}$.

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[2]This is a joint work with V.G.Rau and A.V.Shutov

Theorem 4. *The following conditions are equivalent:*

1) There exists the polycube packing of the set $\{P_j\}$ with packing density $k = \frac{R}{n}$, $R = \sum_{i=1}^{M} r_j$

2) There exists the packing space (L, w) of order n and the set of the vectors $\{x_{0j}\}_{1 \le j \le M}$ such that the points $\{\beta_{ij} + x_{0j}\}_{1 \le i \le r_j, 1 \le j \le M}$ are pairwise different and have pairwise different weights.

Using this theorem we obtain the algorithm which generates all translation packings of a given set of polycube with a given packing density [3].

Every polycube packing can be associated with some d-tuple $(c_1, \dots, c_d)$ with $0 \le c_i \le 2^d - 1$ for $1 \le i \le d$. This d-tuple is called a packing code. We use this code to recognize packings equivalent by some transformation from $SO(d)$. We also use this coding to obtain an algorithm for generation of all periodic polycube tilings with a given volume of fundamental domain and a given number of polycubes.

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Hecke surfaces and Duality transformations in Lattice Spin Systems.

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Abstract

In this paper I discuss some topics which have long interested me . These themes relate with the following subjects:

1. Hecke surfaces and K- regular graphs.

2. Duality transformations for generalized Potts models.

Each of them relates with deep mathematical and physical theories and they have nothing in common at the first sight. However, it become more evident in the last year that a deep internal relations between all these problems exist. Especially interesting and mysterious is the role of Hecke groups in this context.

From this point of view is interesting to study the so called McKay correspondence which attached to any finite group K of $SU(2)$ a certain graph which coincides with affine extensions of Dynkin diagrams . Recently these results were extended by I.Dolgachev to the cocompact discrete subgroups γ of $SU(1,1)$. We consider McKay correspondence for Hecke groups and its relations with two-dimensional conformal field theory.

The second problem which we discuss is the cluster behavior of zeros of the Chromatic Polynomial on graphs.There exists so called "Beraha conjecture"

Conjecture 2. *Let us consider a chromatic polynomial $P_n(q)$ for arbitrary large planar graph. Then the real zeros of $P_n(q)$ cluster round limit points. These limit points are so called"Beraha numbers"$q = [2\cos(\pi/k)]^2$, $k = 2, 3..$*

This conjecture in general is still unproved. There is an interesting approach using quantum groups (H.Saleur). I would like to outline another approach using Hecke graphs.In this case it is necessary to consider the Caley graph generating by Hecke groups. The partition function of Potts anti-ferromagnetic model determined on this graph reduces to the chromatic polynomials with desire properties.

About optimality of Delaunay triangulations[1].

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Let S be a generic set of n points $\{p_1, ..., p_n\}$ in $\mathbb{R}^d$. We denote by DT(S) the Delaunay triangulation of the convex hull of S in $\mathbb{R}^d$ with vertices in S. In [1,2,3] we defined several functionals on the set of all triangulations of S achieving global minimum on DT(S). In this paper we consider two more optimal functionals.

Delaunay triangulation is used in numerous number of applications. It is usually chosen over other triangulations. A logical question may arise: why this triangulation is better than others?

Usually, the advantages of planar Delaunay triangulations are rationalized by the max-min angle criterion [Sibson, 1978]. The sequence of triangle angles, sorted from sharpest to least sharp, is lexicographically maximized over all such sequences constructed from triangulation of S. In particular, the Delaunay triangulation of $S \subset \mathbb{R}^2$ maximizes the minimum angle in any triangle.

The "radius" functional is the mean of circumradii of triangles for planar triangulations [2]. Let t be a triangulation of S in the plane. Assume that each triangle Δ_i of this triangulation is related to the radius R_i of its circumcircle. For every triangulation t is defined the set of circumradii $\{R_1, ..., R_k\}$ of triangles $\Delta_i \in t$. *The functional $\rho(t, a) := \sum R_i^a, a > 0$ attains its minimum if and only if t is the Delaunay triangulation* [2].

The Delaunay triangulation maximizes the arithmetic mean inradius: *The functional $L(t) = \sum r_i$ attains its maximum if and only if t is the Delaunay triangulation* [Lambert, 1994].

For a polygon P its *harmonic index* $\mathrm{hrm}(P) := \sum a_i^2/S(P)$, where $a_1, \ldots, a_m$ are the lengths of sides of P and $S(P)$ is its area. We have: *The harmonic index $hrm(t) := \sum_i hrm(\Delta_i)$ of a triangulation t of $S \subset \mathbb{R}^2$ achieves its minimum if and only if t is the Delaunay triangulation of S* [1,2].

Let t be a triangulation of $S \subset \mathbb{R}^d$. Denote by $R(t, a) := \sum_i R_i^a \, vol(\Delta_i)$.

Conjecture 1: *The functional $R(t, a)$, where $a \geq 1$, achieves its minimum on the set of all triangulations of $S \subset \mathbb{R}^d$ if and only if t is the Delaunay triangulation.*

Let $D(t, a) := \sum_i |b_i - c_i|^a \, vol(\Delta_i)$, where b_i is the barycenter and c_i is the circumcenter of Δ_i.

Conjecture 2: *The functional $D(t, a)$, where $a \geq 2$, achieves its minimum on the set of all triangulations of $S \subset \mathbb{R}^d$ if and only if t is the Delaunay triangulation.*

Theorem 1. *Conjectures 1 and 2 are correct for $d = 2$.*

Theorem 2. *For any $d \geq 2$ the functional $R(t, 2) - D(t, 2)$ attains its minimum if and only if* t =DT(S).

A proof of this theorem follows from the fact that

$$R(t, 2) - D(t, 2) = c(d)Vr(t) + I(S),$$

where $Vr(t)$ (see [2,3]) is a functional which achieves its minimum for t =DT(S), $c(d)$ is a positive constant, and $I(S) > 0$

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On geometry of Peano Curves.

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The lecture present a review of recent results concerning evaluation of square-to-linear ratio and related characteristic of Peano Curves. Different applications of Peano Curves in mathematic and nature will be discussed.

Об ограничении порядка оси паучка в локально правильной системе Делоне[1].

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Пусть в n-мерном пространстве $\mathbb{R}^n$, $\mathbb{H}^n$ или $\mathbb{S}^n$ задано множество точек S, удовлетворяющее следующим двум условиям:

1) расстояние между точками из множества S не меньше r;

2) в замкнутом шаре радиуса R с центром в произвольной точке пространства имеется по меньшей мере одна точка из S.

Тогда множество S называется (r, R)-системой точек или же системой Делоне. Впредь будем считать, что r является наибольшим числом, удовлетворяющим условию 1), а R – наименьшим числом, удовлетворяющим условию 2). Множество точек из S, принадлежащих шару радиуса ρ с центром в точке A из S, обозначим через $S_A(\rho)$. Группу поворотов пространства вокруг точки A, совмещающих $S_A(\rho)$ с собой, обозначим через $H_A(\rho)$. Пусть ρ_1 – наименьшее число, при котором выполнено равенство $H_A(\rho_1 + 2R) = H_A(\rho_1)$. Тогда $S_A(\rho_1 + 2R)$ назовем стабильным множеством точек для A. Совокупность векторов с началом в A и концами во всех остальных точках множества $S_A(\rho_1+2R)$ назовем стабильным паучком точки A и обозначим через $P_A(\rho_1 + 2R)$. В работе [1] доказан следующий критерий:

Критерий. *Если стабильные паучки всех точек из S конгруэнтны, то множество S является правильным и тогда S в целом однозначно задается стабильным паучком $P_A(\rho_1 + 2R)$.*

Система точек S называется правильной, если каждая ее точка одинаково окружена всеми остальными точками из S. Иногда правильная система точек однозначно задается своим предстабильным паучком $P_A(\rho_1)$. Например, вершины платонова разбиения (k^q) составляют правильную систему точек; для нее имеем $\rho_1 = r$ и $H_A(r) = q{\cdot}m$; она однозначно определяется по своему предстабильному паучку $P_A(r)$. (Для архимедовых разбиений это не так [1].) Напомним: при $2(k + q) > kq$ имеются 5 разбиений (k^q) сферы $\mathbb{S}^2$; при $2(k + q) = kq$ имеются 3 разбиения (k^q) евклидовой плоскости $\mathbb{R}^2$, при $2(k + q) < kq$ имеется счетное множество разбиений (k^q) плоскости Лобачевского $\mathbb{H}^2$.

Будем называть (r, R)-систему точек S локально правильной, если для всех точек A из S паучки $P_A(2R)$ конгруэнтны.

Теорема 1. *Если (r, R)-система в $\mathbb{R}^3$ локально правильна, то порядок поворотной оси a из группы $H_A(2R)$ не больше 6.*

Доказательство теоремы 1. От противного. Пусть порядок оси a больше 6. Тогда паучок $P_A(r)$ состоит из направленных вдоль оси a не более чем двух коллинеарных векторов. В самом деле, если бы вектор из паучка $P_A(r)$ не был коллинеарен a, то размножив его поворотами вокруг a, мы получили бы более 6 векторов, принадлежащих $P_A(r)$. Их концы лежали бы на окружности с центром на a. Расстояние между ближайшими концами было бы меньше r, что противоречило бы условию 1).

Итак, векторы паучка $P_A(r)$ коллинеарны, а векторы паучка $P_A(2R)$ составляют [1] базисную совокупность в $\mathbb{R}^3$. Пусть $\rho_\star$ наименьшее из чисел ρ на отрезке $[r, 2R]$, при ко-

[1]Работа выполнена при финансовой поддержке Российского фонда фундаментальных исследований (проект 08-01-00565) и государственной программы ОМН РАН "Современные проблемы теоретической математики".

тором векторы паучка $P_A(\rho_\star)$ неколлинеарны. Тогда $r < \rho_\star \leq 2R$. Пусть B конец того из векторов, который не направлен вдоль a. Тогда вместе с **AB** паучок $P_A(\rho_\star)$ содержит векторы **AC** и **AD**, которые получаются из **AB** при поворотах вокруг a на наименьший угол по и против часовой стрелки соответственно. Точки D, B, C лежат на окружности с центром $N \in a$. Радиусы ND, NB, NC перпендикулярны a. В силу $BD = BC < BN = \sqrt{BA^2 - NA^2} \leq BA = \rho_\star$ векторы **BC** и **BD** принадлежат не только паучку $P_B(\rho_\star)$, но и подпаучку $P_B(\hat{\rho})$, где $\hat{\rho} = BC < \rho_\star$. Так как **BC** и **BD** неколлинеарны, то локальной правильности нет. Противоречие. Следовательно, порядок оси a не больше 6.

Теорема 2. *Если порядок оси a равен* 6, *то из локальной правильности* (r, R)*-системы в* $\mathbb{R}^3$ *следует ее правильность.*

Доказательство теоремы 2. Пусть порядок оси a равен 6. Тогда возможно одно из двух: либо $\rho_\star = r$, либо $\rho_\star > r$. В любом случае паучок $P_A(\rho_\star)$ содержит гексагональную снежинку, однозначно продолжающуюся до плоской гексагональной решетки. В силу этого и локальной правильности вся (r, R)-система в $\mathbb{R}^3$ представляет собой гексагональную решетку или бирешетку.

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Continuous deformation extending over three sphere packing structures: simple cubic lattice, body-centred cubic lattice and face-centred cubic lattice

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Abstract. We show the existence of a continuous deformation extending over three sphere packings corresponding to simple cubic lattice, body-centred cubic lattice and face-centred cubic lattice. Throughout the continuous deformation, each sphere makes contact with at least six spheres, and the entire structure sustains a packing structure. The changes in packing density, contact number and space group under the deformation process are explained in detail.

1 Introduction

Three sphere packings corresponding to simple cubic (SC) lattice, body-centred cubic (BCC) lattice and face-centred cubic (FCC) lattice first appear in the textbook of high-school chemistry. The sphere packing corresponding to SC-lattice has properties: the packing density is about 0.52, the contact number (with surrounding spheres) is 6, and the space group is Pm3m. The sphere packing corresponding to BCC-lattice has properties: the packing density is about 0.68, the contact number is 8, and the space group is Im3m. The sphere packing corresponding to FCC-lattice has properties: the packing density is about 0.74, the contact number is 12, and the space group is Fm3m. All three structures are in the cubic system but belong to the different space groups. Continuous deformation extending over these three sphere packing structures seems have never reported in the past papers.

2 FCC-sphere packing is one of layer stacking structures of hexagonal lattice

Kepler conjectured "Layer stacking of hexagonal lattice is the densest packing of equal spheres" in 1611. Gauss proved it under the periodic packing in 1831. Hales proved it under the general condition in 1998. Thus FCC is one of the densest sphere packing structures. Each layer of hexagonal lattice stacking belongs three kinds of position: A-, B-, or C-site. The stacking of FCC structure is described as the infinite sequence (...ABCABCABCABC...) and this is called ABC-stacking. Standard textbooks of solid state physics in university include the fact the FCC-sphere packing is constructed by the ABC-stacking.

3 SC- and BCC-sphere packings are also ABC-stacking

Each of SC- and BCC-sphere packings can be regarded as the ABC-stacking of hexagonal lattice too. This is an unfamiliar fact to the public but we can confirm the fact by classifying the positions of spheres projected to the plane normal to a <111>-direction. Positions belonging to different heights are classified into A-, B-, or C-site. In FCC-sphere packing, a hexagonal lattice is consist of mutually contacted spheres. But in SC- and BCC-sphere packing, spheres in a hexagonal lattice are separated at regular intervals.

4 Intuitive explanation for the continuous deformation

Now all three sphere packing structures have a common property, that is, layer stackings of hexagonal lattice. Therefore, it might be possible that three structures are described comprehensively. Finally, we found the existence of a continuous deformation extending over three sphere packings. An intuitive explanation is given as follows. Starting from FCC, make distance of spheres in each layer a little larger. At the moment, contact number of each sphere changes from 12 to zero. If we compress the structure along stacking direction until layers are contact, the entire structure sustains a packing structure again.

5 Changes in packing density, contact number and space group under the deformation process

We investigated changes in packing density, contact number and space group under the deformation process. An exact expression for the packing density was successfully calculated. We will explain the details in my talk.

6 The Non-characteristic Orbits

There is an unsolved problem in crystallography, that is, "Find all non-characteristic G-orbits for any space group in 3D". In two-dimensional space, general solution for plane groups was obtained [1]. But in three-dimensional space, the problem was partly solved but the solution was limited into the same crystal family [2]. During the continuous deformation, the space group R3m changes to other space groups Fm3m, Im3m, Pm3m which have higher symmetry than R3m. Fm3m, Im3m, Pm3m are in the cubic system but R3m is in the trigonal system. Therefore, this is an example of non-characteristic orbits which extend over the different crystal family. This is the theoretical significance of the present work .

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Creating real 3D models of mathematics

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Abstract. Accurate models of mathematically defined curved surfaces and polyhedra were constructed. Exact shape data were generated on a personal computer (PC) using mathematical or computer-aided design (CAD) software. Then models were constructed by layered manufacturing, which is well suited for curved surfaces. This method is flexible in that the equation parameters and model scale can be changed easily. For polyhedron models, we created wooden models in addition to layered manufacturing models.

1 Introduction

In the past, there was a systematic study on the development of three-dimensional (3D) mathematical models in Germany which was started around 1870 [1]. This required a host of the best mathematicians to work in collaboration with skilled workmen. They developed many wonderful models, but since production ceased, these models are rarely found these days. We have been involved with the 'Research on the recognition of 3D objects by visually handicapped persons and development of 3D geometrical teaching materials' project in Japan since 2006 [2][3]. One of our main aims is to develop teaching materials to enrich the tactile world for the blind. Many models have been developed thus far. Times have changed. The situation today is quite different from that in 1870. Now, we can construct mathematical models without the assistance of the best mathematicians and skilled workmen. In this paper, we describe our 3D models of mathematically defined curved surfaces and polyhedra.

2 Models of Ring, Horn, and Spindle Torus

Consider an ordinary torus, which is a surface with a hole. Let d denote the radius from the centre of the hole (0,0,0) to the centre of the torus tube and r denote the radius of the tube. There are three types of tori depending on the relative values of d and r. The condition $r < d$ corresponds to a ring torus; $r = d$, a horn torus, which is tangential to itself at the point (0,0,0); and $r > d$, a self-intersecting spindle torus. We created seven kinds of torus which includes three kinds of ring torus, one kind of horn torus, and three kinds of spindle torus. A pair of two equally partitioned models were also created for each of seven models. These models are useful for systematic and intuitive learning of three kinds of torus [3].

3 Models of Hula-Hoop Surface

The horn torus and spindle torus do not have a hole. The abovementioned three kinds of torus are regarded as the loci of circular movement of a circle, which is perpendicular to the horizontal plane. If we bend the perpendicular circle backward by 45°, a hole appears at the centre of locus of circular movement despite the relative size $r > d$. The surface is called Hula-Hoop surface [4]. If we consider a semicircular movement (180°) of the inclined circle, we obtain interesting models which are a pair of mirror images.

4 Models of Bohemian Dome

We continues to consider circular movements of a circle. But in this case, a circular movement is performed in a vertical plane. Then, the circle always turns its face towards the vertical direction. As a result, we obtain an unfamiliar but beautiful surface. This surface is called Bohemian dome [1]. Another beautiful model is obtained by considering the elliptical movement of an ellipse [5].

5 Models of Klein Bottle

The Klein bottle is a non-orientable surface. The surface has no distinct 'inner' and 'outer' sides. And the inside space of the bottle is linked to its outside space. But we cannot create such a model in 3D because a self-intersection is unavoidable. The self-intersection is avoidable in 4D. We created two kinds of model for Klein bottle. One is the correct Klein bottle and the other is the incorrect Klein bottle. A pair of equally partitioned models were created for each of Klein bottles.

6 Models of Polyhedron

We created models of regular polyhedra and semi-regular polyhedra. Sixteen Archimedean polyhedra that include both two mirror images and a Miller's solid. We presented their models by layered manufacturing and wooden polyhedra.

7 Models of Crystallographic Structure

We created three kinds of space-filling polyhedron: cube, truncated octahedron, and rhombic dodecahedron. They are corresponding to Voronoi regions for SC (simple cubic), BCC (body-centred cubic) and FCC (face-centred cubic) lattice respectively.

There are innumerable space-filling polyhedra (e.g., rectangular parallelepiped). But these three are special because they are the only one polyhedron from regular polyhedra, quasi-regular polyhedra, and their dual polyhedra respectively.

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К геометрии фуллеренов.

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Фуллерены Џ самые интригующие объекты мира наноразмерных минеральных и биологических структур. Но до сих пор нет последовательного изложения разрозненных сведений о комбинаторной геометрии фуллеренов. Под фуллеренами далее понимаются не только "нобелевские" полиэдрические молекулы C_{60} и C_{70}, но всякий 3-мерный выпуклый простой полиэдр, на котором разрешены только 5- и 6-угольные грани. "Подводную часть айсберга" при доказательстве теорем о фуллеренах явно или неявно составляют соотношение Эйлера $f + v = e + 2$ для любых Џ простых и непростых Џ выпуклых полиэдров и следующее из него равенство $\sum(6 - k)f_k = 12$ для выпуклых простых полиэдров (для непростых полиэдров имеет место неравенство $>$), где f_k Џ число k-угольных граней. Для фуллеренов оно сводится к соотношению $f_5 = 12$ без ограничений на f6 [1]. Большинство теорем о фуллеренах доказывается указанием процедуры, приводящей к построению проекции Шлегеля. Здесь подразумевается другая фундаментальная теорема о том, что проекция может быть "расправлена" в 3-мерный полиэдр, приччм с реализацией его максимальной симметрии. Имея в виду эти оговорки, приведчм корпус теорем о фуллеренах, восполняющий указанный выше пробел.

Теорема о существовании фуллерена C_v для v=20 и любого чч̧тного $v \geqslant 24$. Доказана в [5] и Џ независимо Џ в [11]. C_{20} Џ это додекаэдр, простейший из фуллеренов. Невозможность фуллерена C_{22} доказывается в [11] невозможностью построения его проекции Шлегеля. В [5, p. 745] этот вопрос считается очевидным: "Polyhedra P_1 and Q_1 obviously do not exist". (Здесь C_{22} обозначен как P_1.) Между тем, весьма досадно то обстоятельство, что невозможность фуллерена C_{22} не удачтся доказать алгебраически, исходя из известных комбинаторно-геометрических соотношений. Существование бесконечной серии фуллеренов начиная с C_{24} в обоих случаях доказывается конструктивно Џ предъявлением "полусферических" фрагментов различной конструкции, композиция которых вместе с различным числом поясов, состоящих из гексагонов, обеспечивает существование фуллерена C_v с нужным $v \geqslant 24$. В [5] приведены 4 "полусферических" фрагмента, в [11] Џ 5, что составляет их полное число. Встраивание в структуру поясов гексагонов порождает серию фуллеренов с собственным названием "тубулены", производство которых на основе углерода весьма важно для электроники.

Теорема о существовании простейших фуллеренов C_v без триад пентагонов, контактирующих в общей вершине, для v=50. Доказана в [7] конструктивным способом с построением двух таких фуллеренов C_{50} (-10m2, 32). Компьютерные перечисления показали, что число таких форм быстро растчт с v, для диапазона C_{50} Џ C_{70} все они найдены и охарактеризованы точечными группами симметрии в [9]. Но отсутствует доказательство того, что такие фуллерены возможны для любого чч̧тного v>50. Еч̧ физической подоплчкой служит то, что в организации таких фуллеренов (по сравнению с фуллеренами с триадами контактирующих пентагонов) совершается важный скачок на пути к их потенциальной стабильности. Есть факты, говорящие о том, что они могут быть стабильными, в особенности при наличии допирующих атомов.

Теорема о существовании фуллерена C_v без контактирующих пентагонов для v =60 и любого чч̧тного $v \geqslant 70$. Доказана в [6, 10] конструктивным способом, по аналогии с доказательством теоремы о существовании фуллерена. Но в [6] использованы 4 "полусферических" фрагмента, тогда как в [10] Ц все 18, заполняющих тот же контур и порождающих гораздо большее разнообразие бесконечных серий фуллеренов без контактирующих пентагонов. Физическая подоплчка теоремы состоит в том, что наиболее стабильны именно фуллерены без контактирующих пентагонов. Эта теорема указывает важные ограничения на число вершин (атомов) таких фуллеренов.

Теорема о существовании икосаэдрических фуллеренов C_v при $v = 20(h^2+hk+k^2)$, где $0 < h \geqslant k \geqslant 0$ Ц целые числа. Доказана в [4]. Важность теоремы состоит в том, что она указывает необходимое и достаточное условие для числа вершин икосаэдрических (самых симметричных и потому потенциально наиболее стабильных) фуллеренов. Достаточность реализуется через конструктивную схему построения фуллерена с заданным v. Можно показать, что фуллерены (h, 0) и (h, h) имеют симметрию -3-5m, фуллерены (h, k) при $h \neq k$ Ц симметрию 235. Биологическая подоплчка теоремы состоит в существовании обширного класса икосаэдрических вирусов, радиолярий и простейших водорослей, для которых теорема указывает строгие приципы классификации структур [2].

Теорема о фуллеренах-генераторах. Доказана в [3]. Показано, что в множестве икосаэдрических фуллеренов существуют бесконечные серии двух типов. (*) Порождается переходом к дуальному полиэдру и усечением его по всем вершинам: $(h, k) \rightarrow (h+2k, h-k)$. Число вершин фуллерена увеличивается при этом в 3 раза. (**) Порождается "преобразованием подобия" $(h, k) \rightarrow (th, tk)$, где t Ц любой натуральный множитель. Число вершин фуллерена увеличивается при этом в t2 раз. Двукратное применение процедуры (*) равносильно процедуре (**) с t=3. Генераторами названы фуллерены, не получаемые процедурами (*) и (**) из более простых. Показано, что генераторами являются те и только те фуллерены (h, k), для которых $h \neq k(mod 3)$. Описание многообразия икосаэдрических форм на уровне генераторов проще, чем на уровне индивидуальных форм. Эта теорема углубляет предыдущую и также имеет отношение к описанию многообразий икосаэдрических вирусов и радиолярий (Circogonia icosahedra, Circogonia dodecahedra и др.).

Теорема об икосаэдрических фуллеренах-изомерах. Анализ икосаэдрических фуллеренов обнаруживает изомеры, простейшие из них: (7, 0) и (5, 3) с 980 вершинами, (9, 1) и (6, 5) с 1820 вершинами. Компьютерными перечислениями найдены простейшие серии до 10 изомеров. В атомном представлении они столь огромны, что имеют лишь теоретический интерес. То есть, в ближайшей области спектра пара чисел (h, k) фиксирует даже комбинаторный тип фуллерена. Но теоретически интересен вопрос о простейших тройках, четвчрках Ж n-ках икосаэдрических фуллеренов-изомеров. Теоретико-числовая задача состоит в отыскании последовательности натуральных N, допускающих заданное число n различных представлений в виде неполного квадрата $h^2 + hk + k^2$. В общем виде она не решена. Легко показать, что в серии икосаэдрических изомеров лишь один фуллерен может иметь симметрию -3-5m. Действительно, икосаэдрические (-3-5m) фуллерены представлены лишь сериями вида $(h, 0)$ и (h, h) с числами вершин h^2 и $3h^2$, соответственно. Очевидно, серии не пересекаются. Но в каждой серии пара (h, k) определяет комбинаторный тип фуллерена однозначно, чем и заканчивается доказательство.

Теорема о замкнутом контуре. Доказана в [10] в виде леммы, предваряющей доказательство теоремы о существовании фуллерена C_v без контактирующих пентагонов (п. 3). Теорема показывает, что число пентагонов внутри любого замкнутого контура на поверхности фуллерена строго определено самим контуром: $f_5 = 6 + e_{in}e_{out}$, где e_{in} и e_{out} Ц числа ребер, примыкающих к контуру изнутри и снаружи, соответственно. Она дачт возможность алгоритмического поиска пентагонов на как угодно большой поверхности фуллерена.

Теорема о среднем радиусе фуллерена C_v. Доказана в [8]. Под средним радиусом фуллерена понимается радиус эквиплощадной сферы. Он ограничен радиусами сфер, вписанных в фуллерен и описанных около него. Показано, что средний радиус фуллерена пропорционален длине ребра гексагона, а коэффициент пропорциональности $\varphi(v)$ табулирован для $v = 60 \div 100$. Размер фуллерена чрезвычайно важен ввиду его способности включать атомы и кластеры с образованием эндоэдральных структур, чрезвычайно важных в различных технических применениях и распространчнных в минеральной природе.

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The Delaunay Tessellation Field Estimator for Cosmic Structure.

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We review the analysis of the Cosmic Web by means of an extensive toolset based on the use of Delaunay and Voronoi tessellations. The Cosmic Web is the salient and pervasive foamlike pattern in which matter has organized itself on scales of a few up to more than a hundred Megaparsec. The weblike spatial arrangement of galaxies and mass into elongated filaments, sheetlike walls and dense compact clusters, the existence of large near-empty void regions and the hierarchical nature of this mass distribution are three major characteristics of the comsic matter distribution.

First, we describe the Delaunay Tessellation Field Estimator. Using the unique adaptive qualities of Voronoi and Delaunay tessellations, DTFE infers the density field from the (contiguous) Voronoi tessellation of a sampled galaxy or simulation particle distribution and uses the Delaunay tessellation as adaptive grid for defining continuous volume-filling fields of density and other measured quantities through linear interpolation. The resulting DTFE formalism is shown to recover the hierarchical nature and the anisotropic morphology of the cosmic matter distribution. The Multiscale Morphology Filter (MMF) uses the DTFE density field to extract the diverse morphological elements - filaments, sheets and clusters - on the basis of a ScaleSpace analysis which searches for these morphologies over a range of scales. Subsequently, we discuss the Watershed Voidfinder (WVF), which invokes the discrete watershed transform to identify voids in the cosmic matter distribution. The WVF is able to determine the location, size and shape of the voids. The watershed transform is also a key element in the SpineWeb analysis of the cosmic matter distribution. Finding its mathematical foundation in Morse theory, it allows the determination of the filamentary spine and connected walls in the cosmic matter density field through the identification of the singularities and corresponding separatrices. The first results of a direct implementation on the Delaunay tessellation itself are presented. Finally, we describe the concept of Alphashapes for assessing the topology of the cosmic matter distribution.