# Tiling by rectangles and discrete inverse problems

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# Statement of the problem

- **Example.** A rectangle $m \times n$, where $m, n \in \mathbb{Z}$, can be tiled by $m \cdot n$ equal squares.
- **Theorem (Dehn, 1903).** *If a rectangle can be tiled by squares (not necessarily equal) then the ratio of the rectangle is rational.*
- **Definition.** *Ratio* of a rectangle = horizontal side/vertical one.
- **Shape Tiling Problem (Keating–King, 2000).** Which polygons can be tiled by rectangles of given ratios (but arbitrary sizes)?
- **Remark.** For *signed* tilings — solved by Keating–King.

# Tiling of a rectangle by rectangles

- **Theorem 1 (Prasolov-S., 2009).** *Suppose that a rectangle of ratio $c$ can be tiled by rectangles of ratios $c_1, \dots, c_n$. Then $c = C(c_1, \dots, c_n)$ for some rational function $C(z_1, \dots, z_n)$ such that*
  - $C(z_1, \dots, z_n) \in \mathbb{Q}(z_1, \dots, z_n)$;
  - $C(tz_1, \dots, tz_n) = tC(z_1, \dots, z_n)$;
  - if $Re\, z_1, \dots, Re\, z_n > 0$ then $Re\, C(z_1, \dots, z_n) > 0$.
- **Problem.** Is the converse theorem true?
- **Remark.** Case $n = 2$ is equivalent to $1/2$ of a theorem of Freiling–Rinne–Laszkovich–Szekeres (1997).
- **Remark.** The result is not *algorithmic* unless $c_1, \dots, c_n$ are algebraic independent.

- **Theorem 2 (Freiling–Laczkovich–Rinne–Szekeres, 1995).** For a number $c > 0$ the following 3 conditions are equivalent:
  - a square can be tiled by rectangles of ratios $c$ and $1/c$;
  - the number $c$ is algebraic and all its algebraic conjugates have positive real parts.
  - for certain positive rational numbers $d_1, \dots, d_m$ we have

$$d_1 c + \cfrac{1}{d_2 c + \dots + \cfrac{1}{d_m c}} = 1.$$

- **Remark.** A new, "physical" proof of this result is obtained.

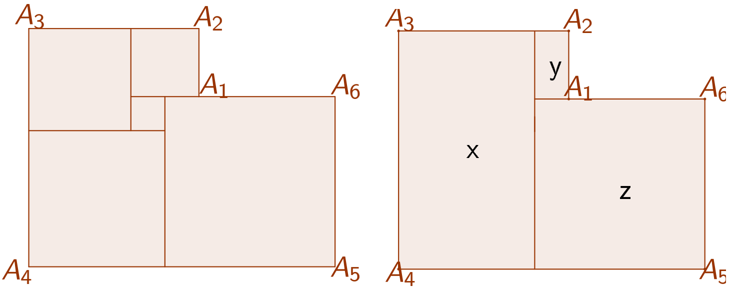
- **Theorem 3 (Prasolov–S., 2009).** For a number $c > 0$ the following 3 conditions are equivalent:
  - a rectangle of ratio $c$ can be tiled by rectangles of ratios $c$ and $1/c$ (in such a way that there is at least one rectangle of ratio $1/c$ in the tiling);
  - the number $c^2$ is algebraic and all its algebraic conjugates distinct from $c^2$ are negative real numbers.
  - for certain positive rational numbers $d_1, \dots, d_m$ we have

$$\cfrac{1}{d_1 c + \cfrac{1}{d_2 c + \dots + \cfrac{1}{d_m c}}} = c.$$

- **Theorem 4 (Kenyon, 1998)** *A hexagon $A_1A_2A_3A_4A_5A_6$ with right angles and one nonconvex angle $A_1$ can be tiled by squares if and only if the system*

$$\begin{cases} A_3A_4 \cdot x + A_1A_2 \cdot y & = A_2A_3, \\ A_5A_6 \cdot z - A_1A_2 \cdot y & = A_6A_1; \end{cases}$$

has a positive rational solution $x, y, z$.

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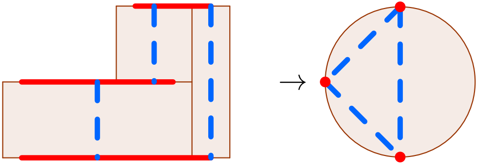
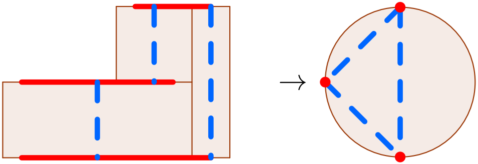
# Tiling of polygons by squares

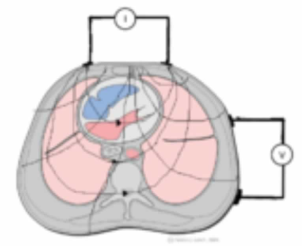
- **Definition.** $\Omega_b$ is the set of all the real $b \times b$ matrices $C_{uv}$ satisfying the following properties:
  - the matrix $C_{uv}$ is symmetric;
  - the sum of the entries of the matrix $C_{uv}$ in each row is zero;
  - for each circularly ordered sequence $(p_1, \ldots, p_{2k})$ the following inequality holds: $(-1)^k \det\{C_{p_i p_{2k-j+1}}\}_{i,j=1}^k \geq 0$.
- **Theorem 5 (S., 2009).** *Let $P$ be a generic orthogonal polygon with $b$ horizontal sides having signed lengths $l_1, \ldots, l_b$ and y-coordinates $U_1, \ldots, U_b$. Then the following 2 conditions are equivalent:*
  - the polygon $P$ can be tiled by squares;
  - there is a matrix $C_{uv} \in \Omega_b$ with rational entries such that $l_v = \sum_{u=1}^b C_{uv} U_u$ for each $v = 1, \ldots, b$.
- **Remark.** This result is not algorithmic unless $U_1, \ldots, U_b$ are linearly independent over $\mathbb{Q}$.

# Physical interpretation

- tiling of a polygon ⇝ electrical resistor circuit;
- maximal horizontal cut ⇝ vertex;
- rectangle ⇝ resistor;
- height of a cut ⇝ voltage;
- signed length of a horizontal side ⇝ current;
- ratio ⇝ conductance;
- touching condition ⇝ **Kirchhoff current law**.
- 

- **Lemma (S.,2009)** Let $P$ be a generic orthogonal polygon with horizontal sides of signed lengths $I_1, \dots, I_b$ and $y$-coordinates $U_1, \dots, U_b$. Then the following 2 conditions are equivalent:
  - the polygon $P$ can be tiled by $m$ rectangles of ratios $c_1, \dots, c_m$;
  - there is a planar electrical network with $b$ boundary vertices, $m$ essential edges of conductances $c_1, \dots, c_m > 0$, incoming voltages $U_1, \dots, U_b$ and incoming currents $I_1, \dots, I_b$.
- **Remark.** For $b = 2$ proved by Brooks–Smith–Stone–Tutte, 1940, cf. Kenyon, 1998, Benjamini–Schramm, 1996.
- 

- **Definition.** *Effective conductance* $C = I_1/(U_1 - U_2)$, if $b = 2$. *Response* — matrix of the map $(U_1, \dots, U_b) \mapsto (I_1, \dots, I_b)$.
- **Lemma.** *Suppose that an electrical network consists of $m$ edges of conductances $c_1, \dots, c_m$. Then the conductance of the network $C(c_1, \dots, c_m)$ has the following properties*:
  - $C(c_1, \dots, c_m) \in \mathbb{Q}(c_1, \dots, c_m)$;
  - $C(c_1, \dots, c_m)$ is degree 1 homogeneous;
  - *Energy conservation law.* If $Re\, c_1, \dots, Re\, c_m > 0$ then $Re\, C(c_1, \dots, c_m) > 0$.
  - *The Rayleigh monotonicity law.* If $c_1, \dots, c_m > 0$ then $\frac{\partial}{\partial c_j} C(c_1, \dots, c_m) \geq 0$;
- **Definition.** In an *alternating-current network* all $c_k \in \mathbb{C}$, $Re\, c_1 > 0$ ($\Leftrightarrow$ positive *heat power*).

- **Electrical Impedance Tomography Problem.** Construct an electrical network with given effective conductance/ response.
- **Remark.** For *direct-current* networks solved by Curtis, Morrow, Colin de Verdiere. For *alternating-current* — open!
- **Remark.** *Continuous* version partially solved by Calderon, Sylvester, Uhlmann, . . .
-  

# "Physical" proofs of Theorems 2 and 5

- **Theorem (Foster–Cauer, 1926).** *For any odd function* $C(\omega) \in \mathbb{Q}(\omega)$ *such that* $\mathrm{Re}\, C(\omega) > 0$, *if* $\mathrm{Re}\, \omega > 0$, *there is an electrical network with edge conductances* $\omega$ *and* $1/\omega$, *and effective conductance* $C(\omega)$.
- **Corollary.** Theorem 2. **Proof.** $C(\omega) := \frac{P(-\omega)-P(\omega)}{P(-\omega)+P(\omega)}$, where $P(\omega)$ is a minimal polynomial of $c$.
- **Theorem (Curtis–Morrow, 2000).** *The set of all possible responses of planar electrical networks with* $b$ *boundary vertices and positive edge conductances is the set* $\Omega_b$.
- **Corollary.** Theorem 5.

# Acknowledgements

THANKS!