On the cohomological rigidity of toric hyperKähler manifolds

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Cohomological rigidity problems

Problem (Cohomological Rigidity Problem)

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E.g., "the Poincaré homology sphere" and "the standard sphere".

However

If we restrict the class of the manifolds, the answer is sometimes affirmative.

Example which satisfies the cohomological rigidity

Definition

We say the quotient manifold

$$H_k = S^3 \times_{S^1} \mathbb{P}(\mathbb{C}_k \oplus \underline{\mathbb{C}})$$

the Hirzebruch surface, where $\mathbb{C}_k$ is the representation space $\mathbb{C}$ with k times rotated S^1-action for $k \in \mathbb{Z}$.

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Remark

H_k is the projectivization of the sum of two line bundles over $\mathbb{C}P^1$, i.e., H_k is a $\mathbb{C}P^1$-bundle over $\mathbb{C}P^1$.

How to prove the cohomological rigidity of H_k

Theorem (Hirzebruch 1951)

$H_k \cong H_{k+2}$, *i.e., their topological types are at most*

$$\begin{aligned} H_0 &= \mathbb{C}P^1 \times \mathbb{C}P^1 \textit{ or} \\ H_1 &= S^3 \times_{S^1} P(\mathbb{C}_1 \oplus \mathbb{C}). \end{aligned}$$

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Proof.

By comparing their cohomology rings

$$\begin{aligned} H^*(H_0) &\simeq \mathbb{Z}[x,y]/\langle x^2, y^2\rangle, \\ H^*(H_1) &\simeq \mathbb{Z}[x,y]/\langle x^2, y(y+x)\rangle. \end{aligned}$$

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In this talk, we study this problem for **toric hyperKähler manifolds**.

Toric hyperKähler manifolds

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Then

The hyperKähler moment map

$$\mu_{\mathbb{R}} \oplus \mu_{\mathbb{C}} : \mathbb{H}^m \to (\mathfrak{t}^m)^* \oplus (\mathfrak{t}^m_{\mathbb{C}})^*$$

can be defined as

$$\begin{aligned} (\mu_I =)\mu_{\mathbb{R}}(z, w) &= \frac{1}{2}\sum_{i=1}^{m}(|z_i| - |w_i|)\partial_i \in (\mathfrak{t}^m)^*; \\ (\mu_J + \sqrt{-1}\mu_K =)\mu_{\mathbb{C}}(z, w) &= 2\sqrt{-1}\sum_{i=1}^{m}(z_i w_i)\partial_i \in (\mathfrak{t}^m_{\mathbb{C}})^*. \end{aligned}$$

Constructive definition by hyperKähler quotient

Moreover

For a subgroup $K \overset{\iota}{\hookrightarrow} T^m$, we also have the hyperKähler moment map of the restricted K-action on $\mathbb{H}^m$ by

$$\mu_{HK} : \mathbb{H}^m \to \mathfrak{k}^* \oplus \mathfrak{k}^*_{\mathbb{C}}$$

by $\mu_{HK} = (\iota^* \oplus \iota^*_{\mathbb{C}}) \circ (\mu_{\mathbb{R}} \oplus \mu_{\mathbb{C}})$.

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Definition

We say the hyperKähler quotient for $\alpha \neq 0 (\in \mathfrak{k}^*)$

$$M_\alpha = \mu_{HK}^{-1}(\alpha, 0)/K$$

a **toric hyperKähler variety**.

Example

Let $K = \Delta$ be the diagonal subgroup in T^{n+1}.
The moment map $\mu_{HK} =: \mathbb{H}^{n+1} \to \mathbb{R} \oplus \mathbb{C}$ is defined by

$$\mu_{HK}(z, w) = \frac{1}{2} \sum_{i=1}^{n+1} (|z_i| - |w_i|) \oplus 2\sqrt{-1} \sum_{i=1}^{n+1} (z_i w_i).$$

Let $\alpha = 1 \in \mathbb{R}$. It is easy to show that

$$M_1 = \mu_{HK}^{-1}(1, 0)/\Delta = T^*\mathbb{C}P^n$$

with the induced $T^n = T^{n+1}/\Delta$ action on $\mathbb{C}P^n$.

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- *Let* X *be a toric manifold. Its cotangent bundle* T^*X *is an open submanifold for some toric hyperKähler orbifold* M.

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- *Let* X *be a toric manifold. Its cotangent bundle* T^*X *is an open submanifold for some toric hyperKähler orbifold* M.
- T^*X *is a toric hyperKähler manifold* $\overset{iff}{\Longleftrightarrow}$ $X = \prod_j \mathbb{C}P^{n_j}$.
- *Its topological structure is detemined by the combinatorial data of a hyperplane arrangement.*

Hyperhamiltonian structure

(M_α, T^n) is hyperhamiltonian, i.e., this action preserves the hyperKähler structure, and there is a hyperKähler moment map $\widetilde{\mu}_{\widehat{\alpha}} = \widetilde{\mu}_{\mathbb{R}} \oplus \widetilde{\mu}_{\mathbb{C}} : M_\alpha \to (\mathfrak{t}^n)^* \oplus (\mathfrak{t}^n_{\mathbb{C}})^*$ such that

$$\widetilde{\mu}_{\mathbb{R}}[z, w] = \frac{1}{2}\sum_{i=1}^{m}(|z_i| - |w_i|)\partial_i - \widehat{\alpha} \in \ker \iota^* \simeq (\mathfrak{t}^n)^* \subset (\mathfrak{t}^m)^*;$$

$$\widetilde{\mu}_{\mathbb{C}}[z, w] = 2\sqrt{-1}\sum_{i=1}^{m}(z_i w_i)\partial_i \in \ker \iota^*_{\mathbb{C}} \simeq (\mathfrak{t}^n_{\mathbb{C}})^* \subset (\mathfrak{t}^m_{\mathbb{C}})^*,$$

where $\widehat{\alpha} \in (\mathfrak{t}^m)^*$ such that $\iota^*(\widehat{\alpha}) = \alpha$.

Summary

A **lift** $\widehat{\alpha} \in (\mathfrak{t}^m)^*$ **of** $\alpha \in \mathfrak{k}^*$ determines **a hyperKähler moment map** on M_α.

Equivalence relations

Let $(M_\alpha, T^n, \widetilde{\mu}_{\widehat{\alpha}})$ and $(M_\beta, T^n, \widetilde{\mu}_{\widehat{\beta}})$ be $4n$-dim toric hyperKähler manifolds with hyperKähler moment maps.

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Definition

We say $(M_\alpha, T^n, \widetilde{\mu}_{\widehat{\alpha}})$ and $(M_\beta, T^n, \widetilde{\mu}_{\widehat{\beta}})$ are weakly isomorphic if there is a weak T^n-diffeomorphism $f : M_\alpha \to M_\beta$ s.t.

1. f preserves the hyperKähler structure;
2. if $f(xt) = f(x)\varphi(t)$ for $\varphi : T^n \to T^n$, the following diagram is commute:

$$\begin{array}{ccc} M_\alpha & \xrightarrow{f} & M_\beta \\ \widetilde{\mu}_{\widehat{\alpha}} \downarrow & & \downarrow \widetilde{\mu}_{\widehat{\beta}} \\ (\mathfrak{t}^n_{\mathbb{R}\oplus\mathbb{C}})^* & \xleftarrow{\varphi^*} & (\mathfrak{t}^n_{\mathbb{R}\oplus\mathbb{C}})^* \end{array}$$

where $(\mathfrak{t}^n_{\mathbb{R}\oplus\mathbb{C}})^* = (\mathfrak{t}^n)^* \oplus (\mathfrak{t}^n_{\mathbb{C}})^*$.

Equivariant cohomology

In order to state main theorem, we introduce the equivariant cohomology.

Definition

Let (M, T) be a T-space. We say $H^*(ET \times_T M)$ an equivariant cohomology and denote it $H_T^*(M)$.

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Remark

$H_T^*(M)$ is not only ring but also $H^*(BT)$-algebra by

$$\begin{array}{ccc} ET \times_T M & \hookleftarrow & M \\ \pi \downarrow & & \\ BT & & \end{array} \quad \overset{H^*}{\Longrightarrow} \quad \begin{array}{ccc} H_T^*(M) & \to & H^*(M) \\ \pi^* \uparrow & & \\ H^*(BT) & & \end{array}$$

Main theorem 1

Theorem

$(M_\alpha, T, \mu_{\widehat{\alpha}}) \equiv_w (M'_{\alpha'}, T, \mu'_{\widehat{\alpha}'}) \stackrel{\text{iff}}{\Longleftrightarrow}$ *there is a weak algebra isomorphism* $f_T^* : H_T^*(M_\alpha; \mathbb{Z}) \to H_T^*(M'_{\alpha'}; \mathbb{Z})$ *s.t.* $(f_T^*)_{\mathbb{R}}(\widehat{\alpha}) = \widehat{\alpha}'$, *where*

$$(f_T^*)_{\mathbb{R}} : (\mathfrak{t}^m)^* \simeq H_T^2(M_\alpha; \mathbb{R}) \xrightarrow{f_T^*} H_T^2(M'_{\alpha'}; \mathbb{R}) \simeq (\mathfrak{t}^m)^*$$

Definition

We say f_T^* a weak algebra isomorphism, if there is $\varphi : H^*(BT) \stackrel{\simeq}{\to} H^*(BT)$ s.t. the following diagram is commute:

$$\begin{array}{ccc} H^*(BT) & \to & H_T^*(M_\alpha) \\ \varphi \downarrow & & \downarrow f_T^* \\ H^*(BT) & \to & H_T^*(M'_{\alpha'}). \end{array}$$

Main theorem 2

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Two toric hyperKähler manifolds are diffeomorphic $\overset{\text{iff}}{\Longleftrightarrow}$ *their cohomology rings are isomorphic and their dimensions are same.*

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Theorem (Bielawsky 1999)

Let $\mathcal{M}_n$ be the set of all complete, connected, $4n$-dimensional, hyperKähler manifolds with effective, hyperhamiltonian T^n-actions. Then all elements in $\mathcal{M}_n$ are diffeomorphic to toric hyperKähler manifolds, and vice versa.

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Corollary

$\mathcal{M}_n$ *satisfies the cohomological rigidity.*

Remark of the cohomological rigidity theorem

Let $\mathcal{M} = \cup_n \mathcal{M}_n$ be the set of all toric hyperKähler manifolds.
Now, $T^*\mathbb{C}P^n$ and $T^*\mathbb{C}P^n \times \mathbb{H}^\ell$ are elements of $\mathcal{M}$.
It is sasy to show that

$$H^*(T^*\mathbb{C}P^n) \simeq H^*(T^*\mathbb{C}P^n \times \mathbb{H}^\ell);$$

however,

$$T^*\mathbb{C}P^n \cong T^*\mathbb{C}P^n \times \mathbb{H}^\ell \overset{iff}{\Longleftrightarrow} \ell = 0.$$

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Thereofore

$\mathcal{M}$ does not satisfy the cohomological rigidity.

Hyperplane arrangements

To define the toric hyperKähler variety M_α, we need to use the exact sequence

$$(\mathfrak{t}^n)^* \xrightarrow{\rho^*} (\mathfrak{t}^m)^* \xrightarrow{\iota^*} \mathfrak{k}^*,$$

and the non-zero element $\alpha \in \mathfrak{k}^*$.
There is a lift $\widehat{\alpha} \in (\mathfrak{t}^m)^*$ of α, i.e., $\iota^*(\widehat{\alpha}) = \alpha$.

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Definition

The hyperplane arrangement $\mathcal{H}_{\widehat{\alpha}} = \{H_1, \ldots, H_m\}$ is defined by the set of hyperplane

$$H_i = \{x \in (\mathfrak{t}^n)^* \mid \langle \rho^*(x) + \widehat{\alpha}, \mathbf{e}_i \rangle = 0\}$$

where $\mathbf{e}_i$ $(i = 1, \ldots, m)$ is the basis of $\mathfrak{t}^m \simeq \mathbb{R}^m$.

Example

$T^*\mathbb{C}P^2$ is constructed by $\Delta \overset{\iota}{\hookrightarrow} T^3$ and $\alpha = 1 \in \mathfrak{k}^*$. Then

$$\iota^* : (\mathfrak{t}^3)^* \ni (a, b, c) \mapsto a + b + c \in \mathfrak{k}^*$$
$$\rho^* : (\mathfrak{t}^2)^* \ni (x, y) \mapsto (x, y, -x - y) \in (\mathfrak{t}^3)^*.$$

We may take $\widehat{\alpha} = (1, 0, 0) \in (\mathfrak{t}^3)^*$.
Because $H_i = \{(x, y) \in (\mathfrak{t}^2)^* \mid \langle (x, y, -x - y) + (1, 0, 0), \mathbf{e}_i \rangle = 0\}$,

$$H_1 = \{(-1, y) \mid y \in \mathbb{R}\};$$
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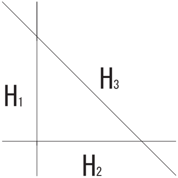

Basic properties of hyperplane arrangements induced by toric hyperKähler manifolds

Proposition (Bielawski-Dancer)

A toric hyperKähler variety $(M_\alpha, T^n, \widetilde{\mu}_{\widehat{\alpha}})$ *is a smooth manifold* $\stackrel{\text{iff}}{\Longleftrightarrow}$ *its hyperplane arrangement* $\mathcal{H}_{\widehat{\alpha}} = \{H_i\}$ *is smooth, i.e.,*

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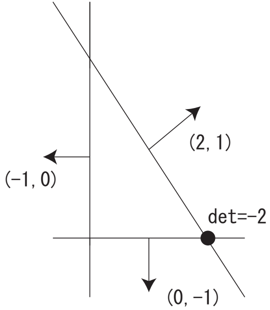
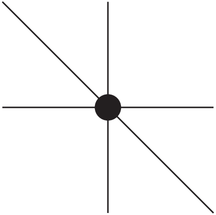
Remark

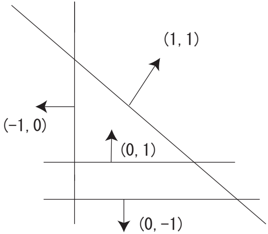
$\rho_*(\mathbf{e}_i) \in \mathfrak{t}^n$ *determines the (weighted) normal vector of* H_i.

Examples of hyperplanes

The left two figures do not appear but the right figure appears as the hyperplane arrangements of toric hyperKähler manifolds.







Fundamental theorems

Theorem (Bielawski-Dancer)

The following two sets are 1:1

1. *Smooth* $(M_\alpha, T^n, \mu_{\widehat{\alpha}})$ *up to hyperhamiltonian.*
2. *Smooth* $\mathcal{H}_{\widehat{\alpha}}$ *up to weighted, cooriented, affine arrangement.*

Theorem (Konno)

Let (M, T) *be a toric hyperKähler manifold and* $\mathcal{H} = \{H_1, \ldots, H_m\}$ *be its hyperplane arrangement. . Then*

$$H_T^*(M; \mathbb{Z}) \simeq \mathbb{Z}[\tau_1, \ldots, \tau_m]/\mathcal{I}$$

where $\deg \tau_i = 2$*, and the ideal* $\mathcal{I}$ *is generated by* $\prod_{j\in J} \tau_j$ *such that* $\cap_{j\in J} H_j = \emptyset$.

Proof 1 –Equivariant cohomological rigidity–

The outline of a proof of the 1^{st} theorem is as follows:

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Step3 For the generator $\tau \in H_T^*(M)$, we can define $Z(\tau)$ called the zero length of τ by the number of $\tau|_p = 0$ for $p \in M^T$.

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- Step3 For the generator $\tau \in H_T^*(M)$, we can define $Z(\tau)$ called the zero length of τ by the number of $\tau|_p = 0$ for $p \in M^T$.
- Step4 If $Z(\tau) = 0$, then $M_\alpha = M'_{\alpha'} \times \mathbb{H}$ for the unique $(4n-4)$-dim toric hyperKähler manifold $M'_{\alpha'}$. Hence, we may regard $Z(\tau) \neq 0$.

Step5

Let $f : H_T^*(M_\alpha) \simeq H_T^*(M_{\alpha'})$ as weak $H^*(BT)$-algebra.
Using the fact that $Z(\tau) = Z(f(\tau))$, we have

$$f : \{\tau_1, \dots, \tau_m\} \to \{\tau_1', \dots, \tau_m'\}$$

up to sign. Therefore, their hyperplane arrangemets are equivalent up to coorinetations.
It follows from the Bielawski-Dancer's theorem that

$$(M_\alpha, T, \mu_{\widehat{\alpha}}) \equiv_w (M'_{\alpha'}, T, \mu_{\widehat{\alpha}'}).$$

Step5

Let $f : H_T^*(M_\alpha) \simeq H_T^*(M_{\alpha'})$ as weak $H^*(BT)$-algebra.
Using the fact that $Z(\tau) = Z(f(\tau))$, we have

$$f : \{\tau_1, \ldots, \tau_m\} \to \{\tau'_1, \ldots, \tau'_m\}$$

up to sign. Therefore, their hyperplane arrangemets are equivalent up to coorinetations.
It follows from the Bielawski-Dancer's theorem that

$$(M_\alpha, T, \mu_{\widehat{\alpha}}) \equiv_w (M'_{\alpha'}, T, \mu_{\widehat{\alpha}'}).$$

Remark

This rigidity is strongly, i.e., $f : H_T^*(M_\alpha) \simeq H_T^*(M_{\alpha'})$ induces the weak isomorphism $(M_\alpha, T, \mu_{\widehat{\alpha}}) \equiv_w (M'_{\alpha'}, T, \mu_{\widehat{\alpha}'})$.

Proof 2 –Cohomological rigidity–

Theorem (Bielawski-Dancer)

The diffeomorphism type of toric hyperKähler manifolds does not depend on the combinatorial structure of their hyperplane arrangements.

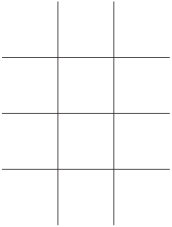
Therefore, by using Proposition about hyperplane arrangements of toric hyperKähler manifolds, the diffeomorphism types of toric hyperKähler manifolds are products of the following two manifolds:

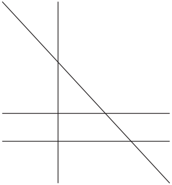
$$M_1(k_1, \ldots, k_n);$$
$$M_2(k_0, k_1, \ldots, k_n),$$

where k_i is the number of hyperplanes which are perpendicular to $\mathbf{e}_i$ $(i = 1, \ldots, n)$ and k_0 is the number of hyperplanes which are perpendicular to $\mathbf{e}_1 + \cdots + \mathbf{e}_n$.

Examples of $M_1(k_1, k_1, \ldots, k_n)$ and $M_2(k_0, k_1, \ldots, k_n)$

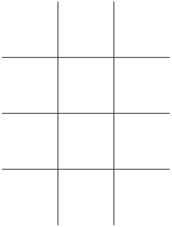
The following left is $M_1(3, 2)$ and the right is $M_2(1, 2, 1)$:

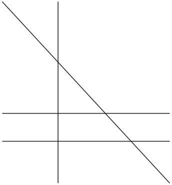




Examples of $M_1(k_1, k_1, \dots, k_n)$ and $M_2(k_0, k_1, \dots, k_n)$

The following left is $M_1(3,2)$ and the right is $M_2(1,2,1)$:





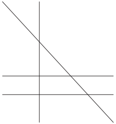
Remark

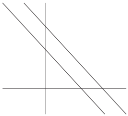
$M_1(k_1, \dots, k_n) = M_1(k_1) \times \cdots \times M_1(k_n)$, *where* $\dim M_1(k_i) = 4$.

Final step of the proof

If $f : H^*(M_1(k_1, \ldots, k_n)) \simeq H^*(M_1(k'_1, \ldots, k'_n))$, then $(k_1, \ldots, k_n) \equiv (k'_1, \ldots, k'_n)$ up to permutation by comparing $Ann(\tau)$ and $Ann(f(\tau))$. (By using the similar argument, we can also prove for the products of M_1's and M_2's.)

For example, the following $M_2(1, 2, 1)$ and $M_2(2, 1, 1)$ have the same cohomology ring:





Therefore, by Theorem (Bielawski-Dancer), we have the 2^{nd} theorem.